On contraction analysis for hybrid systems

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Abstract

Infinitesimal contraction analysis, wherein global asymptotic convergence results are obtained from local dynamical properties, has proven to be a powerful tool for applications in biological, mechanical, and transportation systems. Thus far, the technique has been restricted to systems governed by a single smooth differential or difference equation. We generalize infinitesimal contraction analysis to hybrid systems governed by interacting differential and difference equations. Our theoretical results are illustrated in several examples and applications.

I. Introduction

A dynamical system is *contractive* if all trajectories converge to one another [1]. Contractive systems enjoy strong asymptotic properties, e.g. any equilibrium or periodic orbit is globally asymptotically stable. Provocatively, these global results can sometimes be obtained by analyzing local (or *infinitesimal*) properties of the system's dynamics. In smooth differential (or difference) equations, for instance, a bound on a matrix measure (or induced norm) of the derivative of the equation can be used to prove global contractivity [1], [2], [3], [4], [5], [6]; this approach has been successfully applied to biological [7], [8], [9], mechanical [10], [11], and transportation [12], [13] systems.

At its core, the *infinitesimal* approach to contractivity leverages local dynamical properties of continuous–time flow (or discrete–time reset) to bound the time rate of change of the distance between trajectories. This paper studies infinitesimal contraction analysis generalized to hybrid systems, leveraging local dynamical properties of continuous–time flow *and* discrete–time reset to bound the time rate of change of the distance between trajectories.

Recent work has extended contraction analysis to certain classes of nonsmooth systems. Contraction for systems with a continuous vector field that is piecewise differentiable was first suggested in [14] and rigorously characterized in [15]. Contraction of switched systems, potentially with sliding modes, is studied in [16] by explicitly considering contraction of the sliding vector field and in [17] via a regularization approach that does not require explicit computation of the sliding vector field. The paper [18] considers contraction of Carathéodory switched systems for which the time-varying switching signal is piecewise continuous and allows for different norms for each discrete mode of the switched system.

The present paper complements and, in some cases, extends these prior works by considering a more general class of *hybrid* systems[1] in which time–varying, state–dependent guards trigger instantaneous transitions defined by reset maps between distinct domains. Different norms in each domain are allowed, and domains need not even be of the same dimension. We employ an *intrinsic* distance metric defined in a natural way using these domain–dependent norms such that the distance between a point in a guard and the point it resets to is zero.[2] This approach was proven in [19] to yield a (pseudo[3]) distance metric that assigns finite distance to states in distinct discrete modes (so long as there exist trajectories connecting the modes). The intrinsic distance metric is distinct from the Skorohod [20] or Tavernini [21] *trajectory*

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[1]We exclude Zeno behavior from consideration, and therefore sliding modes are not accommodated here; this exclusion is informed by the applications and examples we seek to study.

[2]i.e. $d(x, \mathcal{R}(x)) = 0$ for all $x \in \mathcal{G}$

[3]On the topological quotient space obtained from the smallest equivalence relation on $\mathcal{D}$ containing $\{(x, \mathcal{R}(x)) : x \in \mathcal{G}\}$, the function is a distance metric compatible with the quotient topology [19, Thm. 13].

metrics [19, Sec. V-A] and from the *distance function* introduced in [22]; it is a particular instantiation of the class of *distance functions* defined in [23].

The conditions we obtain for infinitesimal contraction (Theorem 1 in Section IV) have intuitive appeal: the derivative of the vector field, which captures the infinitesimal dynamics of continuous–time flow, must be infinitesimally contractive with respect to the matrix measure determined by the vector norm used in each discrete mode (34); the *saltation matrix*, which captures the infinitesimal dynamics of discrete–time reset, must be contractive with respect to the induced norm determined by the vector norms used on either side of the reset (35). If upper and lower bounds on *dwell time* are available, we can bound the intrinsic distance between trajectories, regardless of whether this distance is expanding or contracting in continuous– or discrete– time (Corollary 1 in Section IV). Conversely, if the hybrid system is contractive with respect to the intrinsic distance metric (defined in Section III-3), we show that the system is infinitesimally contractive in continuous and discrete time (Theorem 2 in Section VI). We present several examples (in Section IV) and applications (in Section V) to illustrate these theoretical contributions, and conclude with a discussion of our results and possible extensions (Section VII).

II. Notation

Given a collection of sets $\{S_\alpha\}_{\alpha\in A}$ indexed by A, the *disjoint union* of the collection is defined $\coprod_{\alpha\in A} S_\alpha = \bigcup_{\alpha\in A}(\{\alpha\}\times S_\alpha)$. Given $(a,x)\in\coprod_{\alpha\in A} S_\alpha$, we will simply write $x\in\coprod_{\alpha\in A} S_\alpha$ when A is clear from context. For a function γ with scalar argument, we denote limits from the left and right by $\gamma(t^-)=\lim_{\sigma\uparrow t}\gamma(\sigma)$ and $\gamma(t^+)=\lim_{\sigma\downarrow t}\gamma(\sigma)$. Given a smooth function $f: X\times Y\to Z$, we let $D_x f: TX\times Y\to TZ$ denote the *derivative of f with respect to $x\in X$* and $Df=(D_x f, D_y f): TX\times TY\to TZ$ denote the derivative of f with respect to both $x\in X$ and $y\in Y$. Here, TX denotes the *tangent bundle* of X; when $X\subset\mathbb{R}^d$ we have $TX = X\times\mathbb{R}^d$. The *induced norm* of a linear function $M:\mathbb{R}^{n_{j'}}\to\mathbb{R}^{n_j}$ is

$$\|M\|_{j,j'} = \sup_{x\in\mathbb{R}^{n_{j'}}}\frac{|Mx|_j}{|x|_{j'}} \tag{1}$$

where $|\cdot|_j$ and $|\cdot|_{j'}$ denote the vector norms on $\mathbb{R}^{n_j}$ and $\mathbb{R}^{n_{j'}}$, respectively; when the norms are clear from context, we omit the subscripts. The *matrix measure* of $A\in\mathbb{R}^{n\times n}$, denoted $\mu(A)$, is

$$\mu(A) = \lim_{h\downarrow 0}\frac{(\|I+hA\|-1)}{h}. \tag{2}$$

III. Preliminaries

A *hybrid system* is a tuple $\mathcal{H}=(\mathcal{D},\mathcal{F},\mathcal{G},\mathcal{R})$ where:

- $\mathcal{D}$ $=\coprod_{j\in\mathcal{J}}\mathcal{D}_j$ is a set of states where $\mathcal{J}$ is a finite set of discrete states and $\mathcal{D}_j=\mathbb{R}^{n_j}$ is a set of continuous states equipped with a norm $|\cdot|_j$ for some $n_j\in\mathbb{N}$ in each domain $j\in\mathcal{J}$;
- $\mathcal{F}$: $[0,\infty)\times\mathcal{D}\to T\mathcal{D}$ is a time–varying vector field that we interpret as $\mathcal{F}_j=\mathcal{F}|_{[0,\infty)\times\mathcal{D}_j}: [0,\infty)\times\mathcal{D}_j\to\mathbb{R}^{n_j}$ for each $j\in\mathcal{J}$;
- $\mathcal{G}$ $=\coprod_{j\in\mathcal{J}}\mathcal{G}_j$ is a time–varying guard set with $\mathcal{G}_j\subset[0,\infty)\times\mathcal{D}_j$ for all $j\in\mathcal{J}$;
- $\mathcal{R}$: $\mathcal{G}\to\mathcal{D}$ is a time–varying reset map.

If a component such as $\mathcal{F}$, $\mathcal{G}$, or $\mathcal{R}$ is time–invariant, then we suppress time dependence in the corresponding notation.

Before we assess infinitesimal contractivity, we first impose restrictions on the *components* of the hybrid system as well as its *flow*, that is, the collection of trajectories it accepts. To help motivate and contextualize the assumptions, we provide expository remarks following each assumption that explain how each condition is employed in what follows and what specific dynamical phenomena it precludes, and in Section VII-A we discuss why these assumptions are satisfied in several application domains of interest.

1) Hybrid system components and constructions: We begin by stating and discussing assumptions on the hybrid system components.

Assumption 1 (hybrid system components)**.** *For any hybrid system $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$:*

1.1 (vector field is Lipschitz and differentiable) $\mathcal{F}_j = \mathcal{F}|_{\mathcal{D}_j} : [0,\infty) \times \mathcal{D}_j \to \mathcal{D}_j$ is globally Lipschitz continuous and continuously differentiable for all $j \in \mathcal{J}$;

1.2 (discrete transitions are isolated) $\mathcal{R}(t, \mathcal{G}(t)) \cap \mathcal{G}(t) = \emptyset$ for all $t \in [0,\infty)$;

1.3 (guards and resets are differentiable) there exists continuously differentiable and nondegenerate[4] $g_{j,j'} : [0,\infty) \times \mathcal{D}_j \to \mathbb{R}$ such that $\mathcal{G}_{j,j'}(t) \subseteq \{x \in \mathcal{D}_j : g_{j,j'}(t,x) \le 0\} \subseteq \mathcal{G}_j(t)$ and there exists continuously differentiable $\mathcal{R}_{j,j'} : \{(t,x) : x \in \mathcal{D}_j,\ g_{j,j'}(t,x) \le 0\} \to \mathcal{D}_{j'}$ such that $\mathcal{R}_{j,j'}|_{\mathcal{G}_{j,j'}(t)} = \mathcal{R}|_{\mathcal{G}_{j,j'}(t)}$ for each $j, j' \in \mathcal{J}$ and $t \ge 0$ (whenever $\mathcal{G}_{j,j'}(t) \neq \emptyset$);

1.4 (vector field is transverse to guard) $D_t g_{j,j'}(t,x) + D_x g_{j,j'}(t,x) \cdot \mathcal{F}_j(t,x) < 0$ for all $j, j' \in \mathcal{J}$, $t \ge 0$, and $x \in \mathcal{G}_{j,j'}(t)$.

Before we proceed, we make a number of remarks about the preceding Assumption.

Remark 1 (vector fields generate differentiable global flows). Assumption 1.1 ensures there exists a continuously differentiable *flow* $\phi_j : [0,\infty) \times [0,\infty) \times \mathcal{D}_j \to \mathcal{D}_j$ for $\mathcal{F}_j$. In other words, if $x : [\tau,\infty) \to \mathcal{D}_j$ denotes the trajectory for $\mathcal{F}_j$ initialized at $x(\tau) \in \mathcal{D}_j$, then $x(t) = \phi(t,\tau,x)$ for all $t \in [\tau,\infty)$. This condition enables application of classical infinitesimal contractivity analysis for continuous–time flows.

Remark 2 (discrete transitions are isolated without loss of generality). Since Assumption 2.1 below will (in particular) prevent an infinite number of discrete transitions from occurring at the same time instant, Assumption 1.2 is imposed without loss of generality. Indeed, given a hybrid system that permitted at most m discrete transitions at the same instant of time (an example with $m = 2$ can be found in [24, Thm. 8]), the reset map could be replaced with its m–fold composition to yield a hybrid system with isolated discrete transitions that has the same set of trajectories (as defined below).

Remark 3 (guards are closed). Letting $\overline{\mathcal{G}}_{j,j'} = \{x \in \mathcal{D}_j : g_{j,j'}(x) \le 0\}$, and noting that $\overline{\mathcal{G}}_{j,j'}$ is closed by continuity of $g_{j,j'}$, we observe that $\mathcal{G}_j = \bigcup_{j' \in \mathcal{J}} \mathcal{G}_{j,j'} \subset \bigcup_{j' \in \mathcal{J}} \overline{\mathcal{G}}_{j,j'} \subset \mathcal{G}_j$, whence Assumption 1.3 ensures $\mathcal{G}_j = \bigcup_{j' \in \mathcal{J}} \overline{\mathcal{G}}_{j,j'} \subset \mathcal{D}_j$ is a closed set; guard closure is a crucial property used in the definition of the time–of–impact map in the next Remark. (Note that the disjoint components of the guard, $\mathcal{G}_{j,j'}$, are not required to be closed; this affordance will be helpful in the applications considered below.)

Remark 4 (time–of–impact is differentiable). For all $j, j' \in \mathcal{J}$, let the *time–of–impact* $\nu_{j,j'} : [0,\infty) \times \mathcal{D}_j \to [0,\infty]$ for the set $\overline{\mathcal{G}}_{j,j'}$ containing guard $\mathcal{G}_{j,j'}$ be defined by

$$\nu_{j,j'}(\tau, x) = \inf\{t \ge \tau : g_{j,j'}(t, \phi_j(t,\tau,x)) \le 0\}, \tag{3}$$

with the convention that $\inf \emptyset = \infty$. Assumption 1.3 ensures $\nu_{j,j'}$ is well–defined, and Assumption 1.4 ensures $\nu_{j,j'}$ is continuously differentiable wherever it is finite; these properties of the time–of–impact will play a crucial role in the proofs of Propositions 1 and 2 as well as Theorem 1.

Before moving on, we use the time–of–impact maps associated with individual guards to define time–of–impact maps applicable to a discrete mode's entire guard; these maps will subsequently be used to construct the hybrid system's trajectories. For all $j \in \mathcal{J}$, let $\nu_j : [0,\infty) \times \mathcal{D}_j \to [0,\infty]$ be defined by

$$\nu_j(\tau, x) = \min\{\nu_{j,j'}(\tau, x) : j' \in \mathcal{J}\}. \tag{4}$$

Note that ν_j is continuous wherever it is finite, so the sets

$$\mathcal{V}_{j,j'} = \{(\tau, x) \in [0,\infty) \times \mathcal{D}_j : \nu_j(\tau,x) = \nu_{j,j'}(\tau,x)\} \tag{5}$$

are closed for each $j' \in \mathcal{J}$. Letting $\mathcal{V}_j = \bigcup_{j' \in \mathcal{J}} \mathcal{V}_{j,j'} \subset \mathcal{D}_j$, note that $\mathcal{V}_j$ is closed (as a finite union of closed sets) and contains all and only those points in $\mathcal{D}_j$ that flow to a guard. It will be helpful in what follows to define the *time–of–impact* function $\nu : [0,\infty] \times \mathcal{D} \to [0,\infty]$ by $\nu(\tau,x) = \nu_j(\tau,x)$ for all $x \in \mathcal{D}_j$, $j \in \mathcal{J}$.

[4] i.e. $D_x g_{j,j'}(t,x) \neq 0$ for all $t \in [0,\infty)$, $x \in \mathcal{D}_j$

2) Hybrid system trajectories and flow: Informally, a trajectory[5] of a hybrid system is a right–continuous function of time that satisfies the continuous–time dynamics specified by $\mathcal{F}$ and the discrete–time dynamics specified by $\mathcal{G}$ and $\mathcal{R}$. Formally, a function $\chi : [\tau, T) \to \mathcal{D}$ with $\tau \geq 0$ is a *trajectory* of the hybrid system if:

1) $D\chi(t) = \mathcal{F}(t, \chi(t))$ for almost all $t \in [\tau, T)$;
2) $\chi(t^+) = \chi(t)$ for all $t \in [\tau, T)$;
3) $\chi(t^-) = \chi(t)$ if and only if $\chi(t) \notin \mathcal{G}(t)$;
4) whenever $\chi(t^-) \neq \chi(t)$, then $\chi(t^-) \in \mathcal{G}(t)$ and $\chi(t) = \mathcal{R}(t, \chi(t^-))$.

Note that it is allowed, but not required, that $T = \infty$ (although we will shortly impose additional assumptions that ensure trajectories are defined for all positive time). If the domain of χ cannot be extended in forward time to define a trajectory on a larger time domain, then χ is termed *maximal*. The following Proposition ensures that maximal trajectories exist and are unique under the conditions in Assumption 1; its proof is standard [25, Thm. III-1].

Proposition 1 (existence and uniqueness of trajectories)**.** *Under the conditions in Assumption 1, there exists a unique maximal trajectory $\chi : [\tau, T) \to \mathcal{D}$ satisfying $\chi(\tau) = x$ if $x \in \mathcal{D} \backslash \mathcal{G}(\tau)$ or $\chi(\tau) = \mathcal{R}(\tau, x)$ if $x \in \mathcal{G}(\tau)$ for any initial state $x \in \mathcal{D}$ and initial time $\tau \geq 0$.*

Proof. Let $\tau \geq 0$ and $x \in \mathcal{D}_j$; if $x \in \mathcal{G}(\tau)$ then set $x = \mathcal{R}(\tau, x) \notin \mathcal{G}(\tau)$. If $x \in \mathcal{D}_j \backslash \mathcal{V}_j$ for some $j \in \mathcal{J}$ then the trajectory remains within $\mathcal{D}_j$ for all forward time, in which case we let $\chi|_{[\tau,\infty)}(s) = \phi_j(s, \tau, x)$ for all $s \in [\tau, \infty)$. Otherwise, $x \in \mathcal{V}_j$ for some $j \in \mathcal{J}$, so the trajectory flows to $\mathcal{G}(t)$ at time $t = \nu_j(\tau, x) > \tau$, in which case we let $\chi|_{[\tau,t)}(s) = \phi_j(s, \tau, x)$ for all $s \in [\tau, t)$ and set $\chi(t) = \mathcal{R}(t, \phi_j(t, \tau, x))$. Applying the procedure described in the preceding sentences inductively from initial state $\chi(t) \in \mathcal{D} \backslash \mathcal{G}(t)$ at initial time $t \geq 0$ uniquely determines χ at all times on a maximal interval $s \in [\tau, T)$, where $T < \infty$ if and only if the trajectory is *Zeno*, that is, undergoes an infinite number of discrete transitions on the interval $[0, T)$. □

We will restrict the class of trajectories exhibited by the hybrid system in Assumption 2 below. Before imposing these restrictions, we first develop tools that enable analysis of how trajectories vary with respect to initial conditions. At every time $t \geq 0$, the restriction of the reset map $\mathcal{R}_t = \mathcal{R}|_{\mathcal{G}(t)}$ induces an equivalence relation $\overset{\mathcal{R}_t}{\sim}$ on $\mathcal{D}$ defined as the smallest equivalence relation containing $\{(x, y) \in \mathcal{G}(t) \times \mathcal{D} : \mathcal{R}(t, x) = y\} \subset \mathcal{D} \times \mathcal{D}$, for which we write $x \overset{\mathcal{R}_t}{\sim} y$ to indicate x and y are related. The equivalence class for $x \in \mathcal{D}$ is defined as $[x]_{\mathcal{R}_t} = \left\{ y \in \mathcal{D} | x \overset{\mathcal{R}_t}{\sim} y \right\}$. The time–varying *quotient space* induced by the equivalence relation is denoted

$$\mathcal{M}_t = \{[x]_{\mathcal{R}_t} | x \in \mathcal{D}\} \tag{6}$$

endowed with the quotient topology [26, Appendix A]; we note that such quotient spaces have been studied repeatedly in the hybrid systems literature [27], [28], [29], [30], [19].

To define a distance function on the quotient $\mathcal{M}_t$, we will adopt the approach in [19] and use the length of paths that are continuous in the quotient. A path $\gamma : [0, 1] \to \mathcal{D}$ is *smoothly $\mathcal{R}_t$-connected* if: there exists an open set $\mathcal{O} \subset [0, 1]$ such that the closure $\overline{\mathcal{O}} = [0, 1]$; the complement $\mathcal{O}^C$ has (Lebesgue) measure zero; γ is continuously differentiable on $\mathcal{O}$; and $\lim_{r' \uparrow r} \gamma(r') \overset{\mathcal{R}_t}{\sim} \lim_{r' \downarrow r} \gamma(r')$ for all $r \in (0, 1)$ and $\gamma(0) = \lim_{r' \downarrow 0} \gamma(r')$, $\gamma(1) = \lim_{r' \uparrow 1} \gamma(r')$.

A set $\mathcal{O}$ satisfying the above conditions is termed a *support set* for γ at time t. Intuitively, a smoothly $\mathcal{R}_t$-connected path γ is a path through the domains $\{\mathcal{D}_j\}_{j \in \mathcal{J}}$ of the hybrid system that is allowed to jump through the reset map $\mathcal{R}_t$ (forward or backward) and is smooth almost everywhere. With a slight abuse of

[5]Since our analysis concerns infinitesimal contraction in continuous time, we deliberately avoid the concept of an *execution* [25, Def. II.3], which is conventionally defined over a *hybrid time domain*, that is, a set that indexes both continuous and discrete time. Formally, our *trajectory* concept can be regarded as the right–continuous time parameterization (uniquely determined by Assumption 1.2) of the image of the corresponding execution.

notation,[6] we consider γ a path in $\mathcal{M}_t$. With this identification, all $\mathcal{R}_t$–connected paths are (more precisely: descend to) continuous paths in the quotient space $\mathcal{M}_t$. Any support set $\mathcal{O}$ for a smoothly $\mathcal{R}_t$–connected path γ is a countable union of (disjoint) open intervals (cf. [31, Prop. 0.21]); let $\mathcal{O} = \bigcup_{i=1}^{k}(u_i, v_i)$ with possibly $k = \infty$. Because each segment $\gamma|_{(u_i,v_i)}$ is continuously differentiable, the segment is (in particular) continuous, so its image must necessarily belong to a single $\mathcal{D}_j$ for some $j \in \mathcal{J}$.

Assumption 2 (hybrid system flow)**.** *For any hybrid system $\mathcal{H} = (\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$:*

2.1 (Zeno) no trajectory undergoes an infinite number of resets in finite time;

2.2 (continuity of hybrid system flow) with $\phi : \mathfrak{F} \to \mathcal{D}$ denoting the hybrid system flow, i.e. $\phi(t, \tau, x) = \chi(t)$ where[7] $\chi : [\tau, \infty) \to \mathcal{D}$ is the unique trajectory initialized at $\chi(\tau) = x$, the projection $\pi_t \circ \phi(t, \tau, x)$, regarded as a function $\mathcal{D} \to \mathcal{M}_t$, is continuous;

2.3 (invariance of smoothly $\mathcal{R}_t$–connected paths) for all $t \geq \tau \geq 0$ and all smoothly $\mathcal{R}_\tau$-connected paths $\gamma \in \Gamma(\tau)$, $\phi(t, \tau, \gamma(r))$, interpreted as a function of r, is a smoothly $\mathcal{R}_t$-connected path;

2.4 (support sets of smoothly $\mathcal{R}_t$–connected paths) for all $t \geq \tau \geq 0$ and all smoothly $\mathcal{R}_\tau$-connected paths $\gamma \in \Gamma(\tau)$, there exists a support set $\mathcal{O}$ of $\phi(t, \tau, \gamma(\cdot))$ such that, for all $r' \in \mathcal{O}$, there exists $\epsilon > 0$ such that all trajectories $\phi(\cdot, \tau, \gamma(r))$ with $r \in (r' - \epsilon, r' + \epsilon)$ undergo the same sequence of discrete state transitions as $\phi(\cdot, \tau, \gamma(r'))$ on the time interval $[\tau, t]$.

Before we proceed, we make a number of remarks about the preceding Assumption.

Remark 5 (Zeno). Since our results below will strongly leverage the fact that the hybrid system flow is everywhere locally a composition of a finite number of differentiable flows and resets, we cannot easily extend our approach to accommodate Zeno trajectories. (However, we note that approaches other than our own can accommodate *sliding* [16], [17], which is a particular kind of Zeno phenomenon.)

Remark 6 (continuity of hybrid system flow). Trajectories cannot be infinitesimally contractive wherever the flow is discontinuous. To see this, note that the construction of the path–length distance metric d_t below and its compatibility with the topology of the hybrid quotient space $\mathcal{M}_t$ ensures the distance between trajectories on either side of a discontinuity will grow linearly with time.

Remark 7 (support sets of $\mathcal{R}_t$–connected paths). Note that Assumption 1 already suffices to ensure the conditions in Assumptions 2.2–2.3 hold in regions where guards do not “overlap”, i.e. where the intersection of their closures is empty, $\overline{\mathcal{G}}_{j,j'} \cap \overline{\mathcal{G}}_{j,j''} = \emptyset$. Where guards do overlap ($\overline{\mathcal{G}}_{j,j'} \cap \overline{\mathcal{G}}_{j,j''} \neq \emptyset$), Assumption 2.3 does not require that *all* trajectories along a path undergo the same sequence of discrete state transitions, only that the path’s domain contains an open dense subset wherein each connected component undergoes the same sequence of discrete state transitions on finite time horizons; see Fig. 1 for an illustration.

It is well–known [32] that, under favorable conditions, the hybrid system flow ϕ is differentiable almost everywhere and, moreover, its derivative can be computed by solving a jump–linear–time–varying differential equation. The preceding assumptions are favorable enough to ensure the flow has these properties so that, in particular, the derivative along a path can be computed using the jump–linear–time–varying differential equation. These facts are summarized in the following Proposition, whose proof is standard [32].

Proposition 2. *Under Assumptions 1 and 2, given an initial time $\tau \geq 0$ and a smoothly $\mathcal{R}_\tau$-connected path $\gamma \in \Gamma_\tau$, let $\psi(t, r) = \phi(t, \tau, \gamma(r))$ for all $t \geq \tau$ and define*

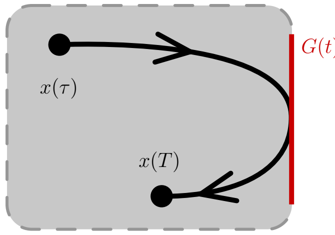
$$w(t, r) = D_r \psi(t, r) \tag{7}$$

whenever the derivative exists. Then $w(\tau^-, r) = D_r \gamma(r)$ and $w(\cdot, r)$ satisfies a linear–time–varying differential equation

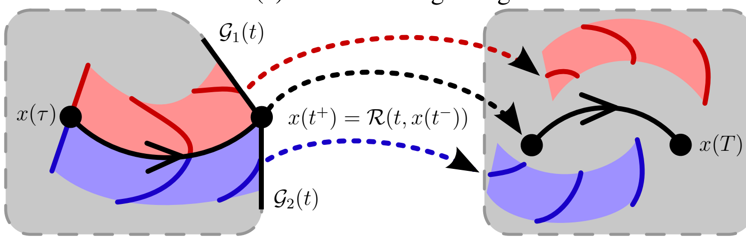
$$D_t w(t, r) = D_x \mathcal{F}(t, \psi(t, r)) w(t, r), \ \ \psi(t^-, r) \in \mathcal{D} \backslash \mathcal{G}(t), \tag{8}$$

[6]Formally, $\pi_t \circ \gamma$ is a path in $\mathcal{M}_t$, where $\pi_t : \mathcal{D} \to \mathcal{M}_t$ is the *quotient projection*.

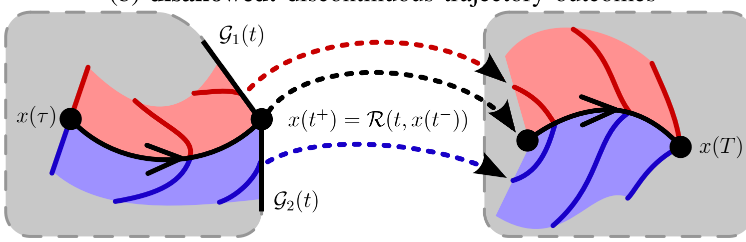
[7]Note that Assumption 2.1 ensures that the maximal time interval in Proposition 1 is $[\tau, \infty)$, i.e. $T = \infty$.



(a) **disallowed:** grazing



(b) **disallowed:** discontinuous trajectory outcomes



(c) **allowed:** overlapping guards with continuous trajectory outcomes

Fig. 1: Illustration of some dynamical phenomena that are (dis)allowed by Assumptions 1 and 2. **Disallowed:** trajectory intersects guard without undergoing reset, i.e. *grazing* (1.4, *vector field is transverse to guard*); trajectory outcomes depend discontinuously on initial conditions (2.2, *continuity of hybrid system flow*). **Allowed:** nonempty intersection of guard closures, i.e. overlapping guards; continuous and piecewise–differentiable trajectory outcomes; different norms used in different discrete modes, which can have different dimensions.

with jumps

$$w(t,r) = \Xi(t, \psi(t^-, r)) w(t^-, r), \;\; \psi(t^-, r) \in \mathcal{G}(t), \tag{9}$$

where $\Xi(t,x)$ *is a* saltation matrix *given by*

$$\Xi(t,x) = D_x\mathcal{R}(t,x) + \frac{(\mathcal{F}_{j'}(t, \mathcal{R}(t,x)) - D_x\mathcal{R}(t,x) \cdot \mathcal{F}_j(t,x) - D_t\mathcal{R}(t,x)) \cdot D_x g_{j,j'}(t,x)}{D_t g_{j,j'}(t,x) + D_x g_{j,j'}(t,x) \cdot \mathcal{F}_j(t,x)} \tag{10}$$

for all $t \geq 0$ *and all* $x \in \mathcal{G}_{j,j'}(t) = \mathcal{G}_j(t) \cap \mathcal{R}^{-1}(\mathcal{D}_{j'})$.

Proof. Given $r \in [0,1]$, suppose that the trajectory χ initialized at $\chi(\tau) = \gamma(r)$ undergoes exactly one discrete transition on the interval $[\tau, t]$ through $\mathcal{G}_{j,j'}(\sigma)$ at time $\sigma \in (\tau, t)$ so that $\psi(\sigma^-, r) = \chi(\sigma^-) \in \mathcal{G}_{j,j'}(\sigma)$. Assumption 2.4 implies that, generically, all trajectories initialized sufficiently close to $\chi(\tau) = \gamma(r)$ also undergo exactly one discrete transition through $\mathcal{G}_{j,j'}$ on the interval $[\tau, t]$. Then for all $s \in [\tau, \sigma)$ we may write $\psi(s,r) = \phi_j(s, \tau, \gamma(s))$, whence $D_s\psi(s,r) = \mathcal{F}_j(s, \psi(s,r))$ and hence

$$\begin{aligned} D_s D_r \psi(s,r) &= D_r D_s \psi(s,r) && (11)\\ &= D_r \mathcal{F}_j(s, \psi(s,r)) && (12)\\ &= D_x \mathcal{F}_j(s, \psi(s,r)) D_r \psi(s,r), && (13) \end{aligned}$$

so (8) is satisfied for times in $[\tau, \sigma)$. Similarly, for all $s \in [\sigma, t]$ we may write

$$\psi(s, r) = \phi_{j'}(s, \nu_{j,j'}(\tau, \gamma(r)), \mathcal{R}(\nu_{j,j'}(\tau, \gamma(r)), \phi_j(\nu_{j,j'}(\tau, \gamma(r)), \tau, \gamma(r))) \tag{14}$$

where $\nu_{j,j'}(\tau, x)$ denotes the time–of–impact for guard $\mathcal{G}_{j,j'}$ for the trajectory initialized at state x at time τ, whence $D_s\psi(s, r) = \mathcal{F}_{j'}(s, \psi(s, r))$ and hence

$$D_s D_r \psi(s, r) = D_r D_s \psi(s, r) \tag{15}$$
$$= D_r \mathcal{F}_{j'}(s, \psi(s, r)) \tag{16}$$
$$= D_x \mathcal{F}_{j'}(s, \psi(s, r)) D_r \psi(s, r), \tag{17}$$

so (8) is satisfied for times in $[\sigma, t]$. Each function that appeared above is continuously differentiable wherever its derivative is evaluated since: Assumption 1.1 implies $\phi_{j'}$ and ϕ_j are continuously differentiable; Assumption 1.3 implies $\mathcal{R}|_{\mathcal{G}_{j,j'}}$ is continuously differentiable; and Assumption 1.4 implies the time–to–impact ν is continuously differentiable.

It remains to be shown that $D_r\psi(\sigma, r)$ is related linearly to $\lim_{s\uparrow\sigma} D_r\psi(s, r)$ via a *saltation matrix* of the form in (10). To see that this is the case, note that

$$\psi(\sigma + \varepsilon, r) = \tag{18}$$
$$\phi_{j'}\left(\sigma + \varepsilon, \nu_{j,j'}(\sigma - \varepsilon, \psi(\sigma - \varepsilon, r)),\right. \tag{19}$$
$$\left.\mathcal{R}(\nu_{j,j'}(\sigma - \varepsilon, \psi(\sigma - \varepsilon, r)), \phi_j(\nu_{j,j'}(\sigma - \varepsilon, \psi(\sigma - \varepsilon, r)), \sigma - \varepsilon, \psi(\sigma - \varepsilon, r)))\right). \tag{20}$$

Differentiating both sides of this expression with respect to r and taking the limit as $\varepsilon \downarrow 0$,

$$D_r\psi(\sigma, r) = [D_2\phi_{j'} \cdot D_2\nu_{j,j'} \tag{21}$$
$$+ D_3\phi_{j'} \cdot (D_1\mathcal{R} \cdot D_2\nu_{j,j'} + D_2\mathcal{R} \cdot [D_1\phi_j \cdot D_2\nu_{j,j'} + D_3\phi_j])] \cdot D_r\psi(\sigma^-, r) \tag{22}$$
$$= \Bigg[D_2\mathcal{R}(\sigma, x) \tag{23}$$
$$+ \Big(\mathcal{F}_{j'}(\sigma, \mathcal{R}(\sigma, x)) - D_2\mathcal{R}(\sigma, x) \cdot \mathcal{F}_j(\sigma, x) - D_1\mathcal{R}(\sigma, x) \Big) \tag{24}$$
$$\times \left(\frac{D_2 g_{j,j'}(\sigma, x)}{D_1 g_{j,j'}(\sigma, x) + D_2 g_{j,j'}(\sigma, x) \cdot \mathcal{F}_j(\sigma, x)} \right) \Bigg] \cdot D_r\psi(\sigma^-, r), \tag{25}$$

where we have made use of the following substitutions that apply when we evaluate both time arguments at σ and the state argument at $x = \psi(\sigma^-, r)$: $D_2\phi_{j'}(\sigma, \sigma, \mathcal{R}(\sigma, x)) = -\mathcal{F}_{j'}(\sigma, \mathcal{R}(\sigma, x))$, $D_1\phi_j(\sigma, \sigma, x) = \mathcal{F}_j(\sigma, x)$, $D_3\phi_{j'}(\sigma, \sigma, \mathcal{R}(\sigma, x)) = I$, $D_3\phi_j(\sigma, \sigma, x) = I$,

$$D_2\nu_{j,j'}(\sigma, x) = \frac{-D_2 g_{j,j'}(\sigma, x)}{D_1 g_{j,j'}(\sigma, x) + D_2 g_{j,j'}(\sigma, x) \cdot \mathcal{F}_j(\sigma, x)}. \tag{26}$$

Thus $D_r\psi(\sigma, r)$ is related linearly to $D_r\psi(\sigma^-, r)$ via a *saltation matrix* of the form in (9), as desired.

Assumption 2.1 ensures there are a finite number of discrete transitions on the (bounded) interval $[\tau, t]$, whence the preceding argument can clearly be applied inductively to accommodate all discrete transitions on the interval $[\tau, t]$. □

3) Hybrid system intrinsic distance: As the final preliminary construction, we define the length of a smoothly $\mathcal{R}_t$–connected path $\gamma : [0, 1] \to \mathcal{D}$ as the sum of the lengths of its segments, and use this *length structure* [33, Ch. 2] to derive a distance metric on $\mathcal{M}_t$. To that end, define the length of a (continuously differentiable) path segment $\gamma|_{(u_i,v_i)} : (u_i, v_i) \to \mathcal{D}_j$ in the usual way using the norm $|\cdot|_j$ in $\mathcal{D}_j$, namely,

$L_j(\gamma|_{(u_i,v_i)}) = \int_{u_i}^{v_i} |D\gamma_j(r)|_j dr$ (we drop the subscript for L when the domain is clear from context), and define length of γ at time t as the sum of the lengths of its segments,

$$L_t(\gamma) = \sum_{i=1}^{k} L(\gamma|_{(u_i,v_i)}) = \int_{\mathcal{O}} |D\gamma(r)|_{j(r)} dr$$
$$= \int_0^1 |D\gamma(r)|_{j(r)} dr, \tag{27}$$

where $j(r) \in \mathcal{J}$ denotes the domain satisfying $\gamma(r) \in \mathcal{D}_{j(r)}$ for each $r \in [0,1]$. (Note that the value of the expression $L_t(\gamma)$ does not depend on the particular support set $\mathcal{O}$, so the length of γ at time t is well–defined.) With $\Gamma(t)$ denoting the set of smoothly $\mathcal{R}_t$-connected paths in $\mathcal{M}_t$, and letting

$$\Gamma(t,x,y) = \{\gamma \in \Gamma(t) : \gamma(0) = x \text{ and } \gamma(1) = y,\ x,y \in \mathcal{D}\} \tag{28}$$

denote the subset of paths that start at $x \in \mathcal{D}$ and end at $y \in \mathcal{D}$, we define the distance $d_t(x,y)$ between x and y at time t by

$$d_t(x,y) = \inf_{\gamma \in \Gamma(t,x,y)} L_t(\gamma), \tag{29}$$

and note that the function $d_t : \mathcal{M}_t \times \mathcal{M}_t \to \mathbb{R}_{\geq 0}$ so defined is a distance metric on $\mathcal{M}_t$ compatible with the quotient topology [19, Thm. 13].

Remark 8 (distance metric). Note that the distance metric defined in (29) is time–varying if (and only if) the guard $\mathcal{G}$ and/or the reset $\mathcal{R}$ are time–varying. Although this property may initially seem counter–intuitive, we argue that it is in fact natural that the intrinsic distance varies with time in hybrid systems that have time–varying guards and/or resets.[8] Indeed, consider a toy example with $\mathcal{D} = \mathcal{D}_1 \coprod \mathcal{D}_2$ where $\mathcal{D}_1 = \mathcal{D}_2 = \mathbb{R}$, i.e. the system state consists of two distinct copies of the real line $\mathbb{R}$, and suppose the state only transitions from $\mathcal{D}_1$ to $\mathcal{D}_2$ through the guard $\mathcal{G}_{1,2}(t) = \{(1,x) \in \mathcal{D}_1 : x = t\}$, at which point the reset function leaves the continuous state unaffected, i.e. $(1,x) \mapsto (2,x)$. Then, according to (29), the distance between state $(1,x) \in \mathcal{D}_1$ and $(2,x) \in \mathcal{D}_2$ equals $d_t((1,x),(2,x)) = 2(x-t)$ for $x > t$ and zero for $x \leq t$. This calculation captures the intuition that the distance between $(1,x)$ and $(2,x)$ decreases as the guard translates up the number line.

IV. Main result

The main contribution of this paper is the provision of local (or *infinitesimal*) conditions under which the distance between any pair of trajectories in a hybrid system (as measured by the intrinsic metric defined in (29)) is globally bounded by an exponential envelope. These conditions are made precise in Theorem 1 and Corollary 1. In the case when the system satisfies a continuous contraction condition within each domain of the hybrid system as well as a discrete nonexpansion condition through the reset map between domains, this exponential envelope is decreasing in time so that the intrinsic distance between trajectories decreases exponentially in time, i.e., the system is contractive.

Example 1. Consider a hybrid system with two domains in the positive orthant of the plane so that $\mathcal{D} = \mathcal{D}_L \coprod \mathcal{D}_R$ with $\mathcal{D}_L = \mathcal{D}_R = \{x \in \mathbb{R}^2 | x_1 \geq 0 \text{ and } x_2 \geq 0\}$, and further take $g_{R,L}(x) = x_1 - 1$ and $g_{L,R}(x) = 1 - x_1$ so that the system is in the left (resp., right) domain $\mathcal{D}_L$ (resp., $\mathcal{D}_R$) when $x_1 < 1$ (resp., $x_1 > 1$). Assume the reset map $\mathcal{R}$ is the identity map and $\dot{x} = \mathcal{F}_j(x) = A_j x$ for $j \in \{L, R\}$ with

$$A_j = \begin{bmatrix} -a_j & 0 \\ 0 & -b_j \end{bmatrix}, \quad a_j, b_j > 0 \quad \text{for } j \in \{L, R\}. \tag{30}$$

All trajectories initialized in $\mathcal{D}$ flow to $\mathcal{D}_L$ and converge to the origin. Equip both domains with the standard Euclidean 2–norm so that $|x|_L = |x|_R = |x|_2$ and consider two trajectories $x(t) = \phi(t, 0, \xi)$,

[8]Of course, time–varying metrics have long been employed in contraction analysis [1].

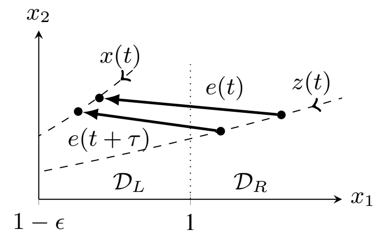


Fig. 2: An illustration of two trajectories $x(t)$ and $z(t)$ of the hybrid system in Example 1 in different domains $\mathcal{D}_L$ and $\mathcal{D}_R$. The distance between trajectories is the Euclidean length of $e(t) = x(t) - z(t)$. When $x(t)$ and $z(t)$ are close, $|e(t)|$ decreases over a short time window $[t, t+\tau]$ if and only if $a_L < a_R$ and $b_L = b_R$, that is, the horizontal component of x decreases at a slower rate than the horizontal component of z and the rates of change of the vertical components are equal.

$z(t) = \phi(t, 0, \zeta)$ with initial conditions $\xi, \zeta \in \mathcal{D}$. Then $d(x(t), z(t)) = |x(t) - z(t)|_2 = |e(t)|_2$ for $e(t) = x(t) - z(t)$. When both trajectories are in the same domain so that $x, z \in \mathcal{D}_j$ for some $j \in \{L, R\}$, the error dynamics obey the dynamics of that domain. It therefore follows that $D_t d(x, z) \leq \max\{-a_j, -b_j\} d(x, z)$ so that the distance decreases at exponential rate $\max\{-a_j, -b_j\}$.

Now suppose x and z are in different domains at some time t and, without loss of generality, assume $x \in \mathcal{D}_L$ and $z \in \mathcal{D}_R$. Writing

$$x = \begin{bmatrix} 1 - \epsilon_L \\ x_2 \end{bmatrix}, \quad z = \begin{bmatrix} 1 + \epsilon_R \\ x_2 + \delta \end{bmatrix} \tag{31}$$

for some $\epsilon_L, \epsilon_R > 0$ and $\delta \in \mathbb{R}$, we have

$$D_t(d(x, z)^2) = D_t((x - z)^T (x - z)) \tag{32}$$

$$= 2(\epsilon_L + \epsilon_R)(a_L - a_R) + 2\delta x_2 (b_1 - b_2) + \text{H.O.T.} \tag{33}$$

where the higher order terms H.O.T. are quadratic in ϵ_L, ϵ_R, and δ. Then $D_t(d(x, z)^2) < 0$ for all $x_2 \geq 0$ and all sufficiently small $\epsilon_L > 0$, $\epsilon_R > 0$, $\delta \in \mathbb{R}$ if and only if $a_L < a_R$ and $b_L = b_R$. In other words, contraction between any two arbitrarily close trajectories transitioning from $\mathcal{D}_R$ to $\mathcal{D}_L$ occurs only if trajectories "slow down" in the direction normal to the guard surface when transitioning domains, and the dynamics orthogonal to the guard are unaffected. This example is illustrated in Fig. 2.

■

We now generalize the intuition of Example 1 to the class of hybrid systems defined in Section III.

Theorem 1. *Under Assumptions 1 and 2, if there exists* $c \in \mathbb{R}$ *such that*

$$\mu_j \left(D_x \mathcal{F}_j(t, x) \right) \leq c \tag{34}$$

for all $j \in \mathcal{J}$, $x \in \mathcal{D}_j \backslash \mathcal{G}_j$, $t \geq 0$, *and*

$$\|\Xi(t, x)\|_{j,j'} \leq 1 \tag{35}$$

for all $j \in \mathcal{J}$, $x \in \mathcal{G}_{j,j'}$, $t \geq 0$, *then*

$$d_t(\phi(t, 0, \xi), \phi(t, 0, \zeta)) \leq e^{ct} d_0(\xi, \zeta) \tag{36}$$

for all $t \geq 0$ *and* $\xi, \zeta \in \mathcal{D}$.

Proof. Given $x(0) = \xi$ and $z(0) = \zeta$, for fixed $\epsilon > 0$, let $\gamma : [0, 1] \to D$ be a piecewise–differentiable $\mathcal{R}_s$-connected path satisfying $\gamma(0) = \xi$, $\gamma(1) = \zeta$, and $L_s(\gamma) < d_0(\xi, \zeta) + \epsilon$, and let $\psi(t, r) = \phi(t, 0, \gamma(r))$. Since $\phi(t, 0, \cdot)$ is piecewise–differentiable, it follows from Assumption 2.2 and 2.3 that $\psi(t, \cdot)$ is a piecewise–differentiable $\mathcal{R}_t$-connected path for all $t \geq 0$. Let $w(t,r) = D_r\psi(t,r)$ whenever the derivative exists. By Proposition 1, $w(t,r)$ satisfies the jump–linear–time-varying equations

$$\dot{w}(t,r) = \frac{\partial f}{\partial x}(t,\psi(t,r))w(t,r),\ \psi(t,r) \in \mathcal{D}\backslash\mathcal{G}, \tag{37}$$

$$w(t^+,r) = \Xi(t,\psi(t^-,r))w(t^-,r),\ \psi(t^-,r) \in \mathcal{G}. \tag{38}$$

We claim that

$$|w(t,r)| \leq e^{ct}|w(0^-,r)| \tag{39}$$

for all $t \geq 0$ and for all $r \in [0,1]$ whenever $w(t,r)$ exists. To prove the claim, for fixed r, let $\{t_i\}_{i=1}^k \subset [0,\infty)$ with $t_0 \leq t_1 \leq \cdots$ and possibly $k = \infty$ be the set of times at which the trajectory $\phi(t,s,\gamma(r))$ intersects a guard so that $\psi(\cdot,r)|_{[t_i,t_{i+1})}$ is continuous for all $i \in \{0,1,\ldots,k-1\}$ where $t_0 = s$ by convention, and, additionally, $\psi(\cdot,r)|_{[t_k,\infty)}$ is continuous if $k < \infty$. Note that if $k = \infty$ then $\lim_{i\to\infty} t_i = \infty$ since Zeno trajectories are not allowed. Now consider some fixed time $T > 0$. If $k < \infty$ and $t_k \leq T$, let $i = k$; otherwise, let i be such that $t_i \leq T < t_{i+1}$. Let j be the active domain of the system during the interval $[t_i,t_{i+1})$, i.e. $\psi(t,r) \in \mathcal{D}_j$ for all $t \in [t_i,t_{i+1})$. With $J(t) = D_x\mathcal{F}_j(\psi(t,r))$ for $t \in [t_i,t_{i+1})$ we have

$$\begin{aligned}
|w(T,r)| &\leq e^{\int_{t_i}^T \mu(J(\tau))d\tau}|w(t_i^+,r)| && (40)\\
&\leq e^{c(T-t_i)}|w(t_i^+,r)| && (41)\\
&\leq e^{c(T-t_i)}\|\Xi(t,\psi(t_i^-,r))\||w(t_i^-,r)| && (42)\\
&\leq e^{c(T-t_i)}|w(t_i^-,r)| && (43)
\end{aligned}$$

where (40) follows by Coppel's inequality applied to (37), (41) follows from (34), (42) follows from (38), and (43) follows from (73); see, e.g., [34, p. 34] for a characterization of Coppel's inequality. Since (40)–(43) holds for any $T < t_{i+1}$, we further conclude that $|w(t_{i+1}^-,r)| \leq e^{c(t_{i+1}-t_i)}|w(t_i^-,r)|$ whenever $i \leq k$. Then, by recursion, $|w(T,r)| \leq e^{cT}|w(s^-,r)|$. Since T was arbitrary, (39) holds.

Again fix $T > 0$. Because $\psi(T,\cdot)$ is a piecewise–differentiable $\mathcal{R}_T$-connected path, there exists a support set $\mathcal{O} = \bigcup_{i=1}^k (u_i,v_i)$ of $\psi(T,\cdot)$ such that $\psi(T,\cdot)|_{(u_i,v_i)}$ is continuously–differentiable for all $i \in \{0,1,\ldots,k\}$. It follows that

$$L_T\left(\psi(T,\cdot)|_{(u_i,v_i)}\right) = \int_{u_i}^{v_i} |w(T,\sigma)|d\sigma. \tag{44}$$

Then

$$\begin{aligned}
L_T(\psi(T,\cdot)) &= \sum_{i=1}^k L_T\left(\psi(T,\cdot)|_{(u_i,v_i)}\right) && (45)\\
&\leq \int_0^1 |w(T,\sigma)|d\sigma && (46)\\
&\leq e^{cT}\int_0^1 |w(s^-,\sigma)|d\sigma && (47)\\
&= e^{cT}L_s(\gamma) && (48)\\
&\leq e^{cT}(1+\epsilon)d_0(\xi,\zeta) && (49)
\end{aligned}$$

where (47) follows from (39), and (48) follows because $w(0^-,r) = D_r\gamma(r)$. In addition, observe

$$d_T(\phi(T,0,\xi),\phi(T,0,\zeta)) \leq L_T(\psi(T,\cdot)). \tag{50}$$

Since T was arbitrary and ϵ can be chosen arbitrarily small, (35) holds. □

Now suppose that uniform upper or lower bounds on the *dwell time* between successive resets are known. Then the number of discrete state transitions is upper or lower bounded on any compact time horizon, and the proof of Theorem 1 can be adapted to derive an exponential bound on the intrinsic distance between any pair of trajectories as in the following Corollary.

Corollary 1. *Under Assumptions 1 and 2, suppose the dwell time between resets is at most* $\overline{\tau} \in (0, \infty]$ *and at least* $\underline{\tau} \in [0, \infty)$*,*

$$\mu_j(D_x \mathcal{F}_j(t, x)) \leq c \tag{51}$$

for some $c \in \mathbb{R}$ *for all* $j \in \mathcal{J}$*,* $x \in D_j \backslash \mathcal{G}_j$*,* $t \geq 0$*, and*

$$\|\Xi(t, x)\|_{j,j'} \leq K \tag{52}$$

for some $K \in \mathbb{R}_{\geq 0}$ *and all* $j \in \mathcal{J}$*,* $x \in \mathcal{G}_{j,j'}$*,* $t \geq 0$*. Then, for all* $t \geq s \geq 0$*,*

$$d_t(\phi(t, 0, \xi), \phi(t, 0, \zeta)) \leq \max\{K^{\lceil t/\underline{\tau} \rceil}, K^{\lfloor (t-s)/\overline{\tau} \rfloor}\} e^{c(t-s)} d_s(\xi, \zeta). \tag{53}$$

In particular, if $\max\{K e^{c\underline{\tau}}, K e^{c\overline{\tau}}\} < 1$ *then*

$$\lim_{t \to \infty} d_t(\phi(t, s, \xi), \phi(t, s, \zeta)) = 0 \tag{54}$$

for all $\xi, \zeta \in \mathcal{D}$*.*

Example 2 (planar piecewise–linear system)**.** Consider a piecewise–linear system with states in the left– and right–half plane,

$$\mathcal{D} = \mathcal{D}_- \coprod \mathcal{D}_+, \ \mathcal{D}_\pm = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \pm x_1 \geq 0 \right\}, \tag{55}$$

whose continuous dynamics are given by $\dot{x} = A_\pm x, \ x \in \mathcal{D}_\pm$, where

$$A_\pm = \begin{bmatrix} \alpha_\pm & -\beta_\pm \\ \beta_\pm & \alpha_\pm \end{bmatrix} \tag{56}$$

so that $\operatorname{spec} A_\pm = \alpha_\pm \pm j\beta_\pm$ and hence the standard Euclidean matrix measure $\mu_2(A_\pm) = \sigma_{\max}\left(\frac{1}{2} A_\pm^\top + A_\pm\right) = \alpha_\pm$. Supposing $\beta_\pm > 0$, all trajectories in $D_\pm$ will eventually reach the set

$$G_\pm = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \pm x_1 \leq 0, \pm x_2 > 0 \right\}, \tag{57}$$

where a reset will be applied that scales the second coordinate by $c_\pm > 0$,

$$\forall x \in G_\pm \subset \mathcal{D}_\pm : x^+ = \mathcal{R}_\pm(x^-) = (x_1^-, c_\pm x_2^-) \in \mathcal{D}_\mp. \tag{58}$$

This yields a saltation matrix

$$\begin{aligned} \Xi_\pm &= D_x \mathcal{R}_\pm + \left(\mathcal{F}^\mp - D_x \mathcal{R}_\pm \cdot \mathcal{F}^\pm \right) \frac{D_x g_\pm}{D_x g_\pm \cdot \mathcal{F}^\pm} \\ &= \begin{bmatrix} \frac{\beta_\mp}{\beta_\pm} & 0 \\ \frac{1}{\beta_\pm} \left(\alpha_\pm c_\pm - \alpha_\mp \right) & c_\pm \end{bmatrix}. \end{aligned} \tag{59}$$

With respect to the standard Euclidean 2–norm:

1) The continuous–time flows are contractive if $\alpha_\pm < 0$, expansive if $\alpha_\pm > 0$.
2) Unless $A_\pm = A_\mp$ and $\mathcal{R}_\pm = \mathrm{id}_{\mathbb{R}^2}$, one of the discrete–time resets is an expansion.

The first claim follows directly from $\mu_2(A_\pm) = \alpha_\pm$. To see that the second claim is true, note that $\beta_+ \neq \beta_-$, $c_+ > 1$, or $c_- > 1$ implies one of the diagonal entries of one of the Ξ's are expansive. Taking $\beta_+ = \beta_-$ and $c_\pm \leq 1$ to ensure that the diagonal entries of $\Xi_\pm$ are non–expansive yields a saltation matrix of the form

$$\Xi = \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} \tag{60}$$

with singular values

$$\sigma(\Xi) = \operatorname{spec} \frac{1}{2} \left(\Xi^\top + \Xi \right) = \frac{1}{2} \left((c+1) \pm \sqrt{d^2 + (c-1)^2} \right); \tag{61}$$

unless $c = 1$ (i.e. $c_+ = c_- = 1$ so $\mathcal{R}_\pm = \mathrm{id}_{\mathbb{R}^2}$) and $d = 0$ (i.e. $\alpha_+ = \alpha_-$ so $A_+ = A_-$), one of these singular values is larger than unity. ■

We now address an important special case, namely, when domains have the same dimension, are equipped with the same norm, and resets are simply translations (e.g., identity resets). The first part of the following Proposition establishes that the induced norm of the saltation matrix is lower bounded by unity; in the particular case of the standard Euclidean 2–norm, the second part of the Proposition shows that the induced norm of the saltation matrix is equal to unity if and only if the difference between the vector field evaluated at x and $\mathcal{R}(x)$ lies in the direction of the gradient of the guard function.

Proposition 3. *Under Assumptions 1 and 2, suppose: the guard set $\mathcal{G}$ and the reset map $\mathcal{R}$ are time-invariant; $\mathcal{R}_{j,j'}$ is a translation for some $j, j' \in \mathcal{J}$ (i.e., $D\mathcal{R}_{j,j'}(x) = I$ for all $x \in \mathcal{G}_{j,j'}$); and $|\cdot|_j = |\cdot|_{j'}$. Then the following properties hold:*

1) *$\|\Xi(t,x)\|_{j,j'} \geq 1$ for all $x \in \mathcal{G}_{j,j'}$, and all $t \geq 0$;*
2) *If $|\cdot|_j = |\cdot|_{j'} = |\cdot|_2$ where $|\cdot|_2$ denotes the standard Euclidean 2–norm, then $\|\Xi(t,x)\|_{j,j'} = 1$ if and only if*

$$\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x) = \alpha(t,x) Dg_{j,j'}(x)^T \tag{62}$$

for all $t \geq 0$ and all $x \in \mathcal{G}_{j,j'}$ for some $\alpha : [0,\infty) \times \mathcal{G}_{j,j'} \to \mathbb{R}$ satisfying

$$0 \leq \alpha(t,x) \leq \frac{-2Dg_{j,j'}(x) \cdot \mathcal{F}(t,x)}{|Dg_{j,j'}(x)|_2^2}. \tag{63}$$

Proof. 1) Under the hypotheses of the proposition, we have that

$$\Xi(t,x) = I + \frac{(\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x)) \cdot Dg_{j,j'}(x)}{Dg_{j,j'}(x) \cdot \mathcal{F}_j(t,x)}. \tag{64}$$

Fix $x \in \mathcal{G}_{j,j'}$ and let $z \in \mathrm{Null}(Dg_{j,j'}(x))$. Then $\Xi(t,x)z = z$ so that always $\|\Xi(t,x)\|_{j,j'} \geq 1$.

2) We first consider the case $\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x) \notin \mathrm{span}\{Dg_{j,j'}(x)^T\}$ for some $t \geq 0$, $x \in \mathcal{G}_{j,j'}$. Choose $z \in \{z : (\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x))^T z = 0 \text{ and } (Dg_{j,j'}(x))z \neq 0\}$. Then

$$\Xi(t,x)z = z + \beta(\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x)) \tag{65}$$

where $\beta = \frac{1}{\dot{g}_{j,j'}(x)} Dg_{j,j'}(x)z$. But then $\Xi(t,x)z$ is the hypotenuse of a right triangle with legs z and $\beta(\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x))$ with nonzero length so that $|\Xi(t,x)z|_2 > |z|_2$ and therefore $\|\Xi(t,x)\|_2 > 1$. Thus we have shown that $\|\Xi(t,x)\|_{j,j'} \leq 1$ implies $\mathcal{F}_{j'}(t, \mathcal{R}(x)) - \mathcal{F}_j(t,x) \in \mathrm{span}\{Dg_{j,j'}(x)^T\}$, *i.e.*, (62) holds for some $\alpha(t,x)$. We now show that, in particular, (63) holds and, moreover, (62)–(63) implies $\|\Xi(t,x)\|_{j,j'} \leq 1$.

Suppose (62) holds. Then, for all $t \geq 0$ and all $x \in \mathcal{G}_{j,j'}$,

$$\Xi(t,x) = I + \frac{\alpha(t,x)}{\dot{g}_{j,j'}(t,x)} Dg_{j,j'}(x)^T Dg_{j,j'}(x) \tag{66}$$

and

$$\|\Xi(t,x)\|_2 \le 1 \iff \|\Xi(t,x)\|_2^2 \le 1 \tag{67}$$

$$\iff \Xi(t,x)^T \Xi(t,x) \preceq I \tag{68}$$

$$\iff (I + \frac{\alpha(t,x)}{\dot{g}_{j,j'}(t,x)} Dg_{j,j'}(x)^T Dg_{j,j'}(x))(I + \frac{\alpha(t,x)}{\dot{g}_{j,j'}(t,x)} Dg_{j,j'}(x)^T Dg_{j,j'}(x)) \preceq I \tag{69}$$

$$\iff \left(I + \left(2\frac{\alpha(t,x)}{\dot{g}_{j,j'}(t,x)} + \alpha(t,x)^2 |Dg_{j,j'}(x)|_2^2\right) Dg_{j,j'}(x)^T Dg_{j,j'}(x)\right) \preceq I \tag{70}$$

$$\iff \left(2\frac{\alpha(t,x)}{\dot{g}_{j,j'}(t,x)} + \alpha(t,x)^2 |Dg_{j,j'}(x)|_2^2\right) \le 0 \tag{71}$$

$$\iff \text{(63) holds.} \tag{72}$$

□

Example 3. A common special class of hybrid systems for which Proposition 3 is applicable is the class of *piecewise–smooth systems* for which the flow is continuous but is governed by a vector field that is only piecewise–smooth. Contraction of such systems has been previously analyzed in [17]. In this example, we show how Theorem 1 and Proposition 3 apply to such systems and compare to results reported in [17].

Consider a hybrid system with two modes $\mathcal{J} = \{-,+\}$ with $\mathcal{D}_- = \mathcal{D}_+ = \mathbb{R}^n$ for some n, time–invariant vector field $\mathcal{F}$, and time–invariant guard sets defined as $\mathcal{G}_{+,-} = \{x : g(x) \le 0\}$ and $\mathcal{G}_{-,+} = \{x : -g(x) \le 0\}$ for some continuously differentiable g. Notice that $\mathcal{D}_\pm \backslash \mathcal{G} = \{x : \pm g(x) > 0\}$. Further suppose $\mathcal{R}(x) = x$ for all $x \in \mathcal{G}$, i.e., identity reset map, and consider $|\cdot|_- = |\cdot|_+ = |\cdot|$ for some norm $|\cdot|$. Here, $\{x : g(x) = 0\}$ is the *switching surface* or *switching manifold*. While we study the case with only two modes, the basic idea extends to a domain partitioned into a collection of disjoint open sets separated by codimension–1 switching surfaces.

Suppose the system satisfies the conditions of Theorem 1 and is therefore contractive. Consider some $x \in \mathcal{G}_{-,+}$ and note that by Assumption 2.1, we must have $Dg(x) \cdot F_+(x) > 0$ so that the trajectory initialized in $\mathcal{D}_-$ at x transitions through the switching surface and flows away from the surface in mode $+$, avoiding a sliding mode condition. We note that sliding modes are allowed in [17] and accommodated with a regularization approach, however, such systems are disallowed in the present analysis by Assumption 1.

Define $\beta(x) = \frac{1}{Dg(x) \cdot \mathcal{F}_-(x)}$ and $M(x) = (\mathcal{F}_+(x) - \mathcal{F}_-(x)) \cdot Dg(x))$, and observe that $\|\Xi(x)\| = \|I + \beta(x)M(x)\|$. Further, note that Proposition 3, part 1 coupled with condition (35) of Theorem 1 implies that, necessarily, $\|\Xi(x)\| = 1$.

We now show that, also, necessarily $\mu(M(x)) = 0$. To see this, consider $\|I + r\beta(x)M(t,x)\|$ for $r \in [0,1]$. Then

$$1 \le \|I + r\beta(x)M(x)\| = \|(1-r)I + r(I + \beta(x)M(x))\| \tag{73}$$

$$\le \|(1-r)I\| + \|r(I + \beta(x)M(x))\| \tag{74}$$

$$= 1 \tag{75}$$

where the first inequality holds by the same argument of Proposition 3 part 1. Therefore, $\|I + hM(x)\| = 1$ for all sufficiently small $h > 0$. Then $\mu(M(x)) = \lim_{h \to 0^+} \frac{1}{h}(\|I + hM(x)\| - 1) = 0$.

Comparing to [17], we see that $\mu(M(x)) = 0$ along with $\mu(D_x \mathcal{F}_\pm(x)) \le c$ for some $c < 0$, *i.e.*, (34) of Theorem 1, are sufficient conditions for contraction as given in [17, Theorem 6]. Thus, the conditions of Theorem 1, when specialized to piecewise–smooth systems with no sliding modes, implies the conditions provided in [17, Theorem 6]. ■

Remark 9 (summary of main result). Our main contribution is Theorem 1, where we generalize infinitesimal contractivity analysis to the class of hybrid systems satisfying Assumptions 1 and 2. This generalization has intuitive appeal, since it combines infinitesimal conditions on continuous–time flow (via the matrix measure of the vector field derivative, (34)) and discrete–time reset (via the induced norm of the saltation matrix, (35)) that parallel the conditions imposed separately in prior work on smooth continuous–time and discrete–time systems, and establishes contraction with respect to the hybrid system's intrinsic distance metric, (36). With bounds on *dwell time*, i.e. the time between discrete transitions, our tools yield a bound in Corollary 1 on the intrinsic distance between trajectories regardless of whether the dynamics are contractive or expansive. Proposition 3 provides specialized results when the reset is simply a translation, a case that relates to prior work on infinitesimal contraction of nonsmooth vector fields.

V. Applications

In this section, we study the implications of our results in two application domains. First, Section V-A considers a hybrid system with identity resets that arises in the study of vehicular traffic flow. Traffic congestion can cause a discontinuous change in the speed of traffic flow on a segment of road, and the speed does not recover immediately as congestion decreases, so the traffic flow exhibits a hysteresis effect; this hysteresis is important for accurately capturing traffic flow patterns and requires a hybrid model of the dynamics. Second, Section V-B presents a class of hybrid systems with non–identity resets that arise when modeling mechanical systems subject to unilateral constraints. Several variations are considered that illustrate application of our approach to systems wherein the state dimension and applicable norm changes through reset.

A. Traffic flow with capacity drop

Consider a length of freeway divided into two segments or *links*. The state of the system is the traffic *density* on the two links. Traffic flows from the first segment to the second. The second link has a finite *jam density* $x_2^{\text{jam}} > 0$, and we consider link 1 to have infinite capacity so that always the state x satisfies $x \in \mathcal{X} = [0,\infty) \times [0, x_2^{\text{jam}}] \subset \mathbb{R}^2$.

The system has two modes, an *uncongested* (resp., *congested*) mode for which the flow between the two links depends only on the density of the upstream (resp., downstream) link. The dynamics of the uncongested mode is

$$\dot{x}_1 = u(t) - \Delta_1(x_1) \tag{76}$$

$$\dot{x}_2 = \Delta_1(x_1) - \Delta_2(x_2) \tag{77}$$

for which we write $\dot{x} = \mathcal{F}_{\text{uncon}}(x,t)$ assuming a fixed $u(t)$, and for the congested mode is

$$\dot{x}_1 = u(t) - S_2(x_2) \tag{78}$$

$$\dot{x}_2 = S_2(x_2) - \Delta_2(x_2) \tag{79}$$

for which we write $\dot{x} = \mathcal{F}_{\text{con}}(x,t)$ where Δ_1 and Δ_2 are continuously differentiable and strictly increasing *demand* functions satisfying $\Delta_1(0) = \Delta_2(0) = 0$, and S_2 is a continuously differentiable and strictly decreasing *supply* function satisfying $S_2(x_2^{\text{jam}}) = 0$; see [12] for further details of the model.

The system is in the congested mode only (but not necessarily) if $\Delta_1(x_1) \geq S_2(x_2)$. Moreover, empirical studies suggest that traffic flow exhibits a hysteresis effect such that traffic remains in the uncongested mode until $x_2 \geq \bar{x}_2$ for some $\bar{x}_2$ and does not return to the uncongested mode until $x_2 \leq \underline{x}_2$ for some $\underline{x}_2 < \bar{x}_2$ [35], [36]. Here, we assume $\bar{x}_2 \in [0, x_2^{\text{crit}})$ where x_2^{crit} is the unique density satisfying $\Delta_2(x_2) = S_2(x_2)$; see Fig. 3. This effect is called *capacity drop*.

We model the traffic flow as a hybrid system with four domains $\mathcal{D}_{\text{SC}}, \mathcal{D}_{\text{S}\bar{\text{C}}}, \mathcal{D}_{\bar{\text{S}}\text{C}}, \mathcal{D}_{\bar{\text{S}}\bar{\text{C}}}$ where

$$\mathcal{D}_{\text{SC}} = \mathcal{X} \cap \{x : \Delta_1(x_1) \leq S_2(x_2)\} \cap \{x : x_2 \geq \underline{x}_2\}, \tag{80}$$

$$\mathcal{D}_{\bar{\text{S}}\text{C}} = \mathcal{X} \cap \{x : \Delta_1(x_1) \geq S_2(x_2)\} \cap \{x : x_2 \geq \underline{x}_2\}, \tag{81}$$

$$\mathcal{D}_{\text{S}\bar{\text{C}}} = \mathcal{X} \cap \{x : \Delta_1(x_1) \leq S_2(x_2)\} \cap \{x : x_2 \leq \bar{x}_2\}, \tag{82}$$

$$\mathcal{D}_{\bar{\text{S}}\bar{\text{C}}} = \mathcal{X} \cap \{x : \Delta_1(x_1) \geq S_2(x_2)\} \cap \{x : x_2 \leq \bar{x}_2\}, \tag{83}$$

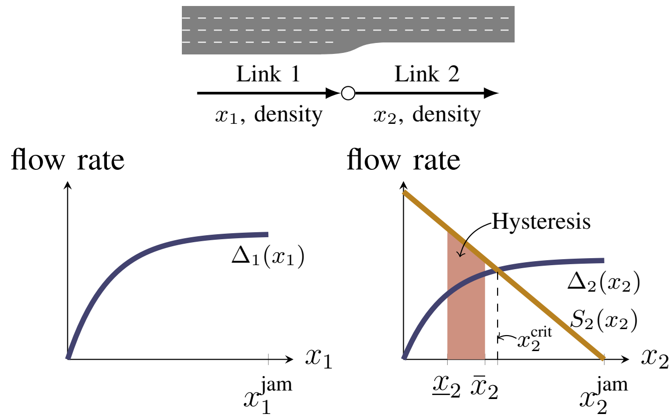


Fig. 3: Traffic flows from link 1 to link 2. Flow at the interface of link 1 and link 2 depends on the demand $\Delta_1(x_1)$ of link 1 and the supply $S_2(x_2)$ of link 2 and exhibits a hysteresis effect. Traffic exits the network at a flow rate equal to the demand $\Delta_2(x_2)$ of link 2.

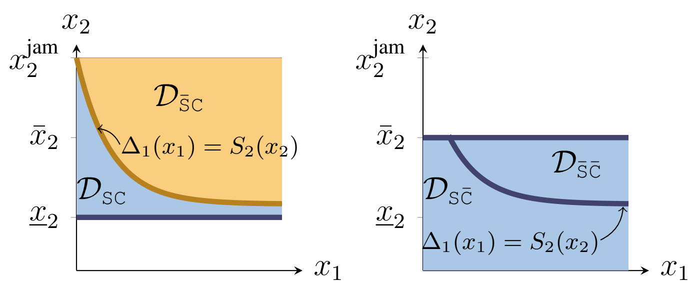


Fig. 4: The traffic network is modeled as a hybrid system with four domains, $\mathcal{J} = \{\mathtt{S}\bar{\mathtt{C}}, \bar{\mathtt{S}}\bar{\mathtt{C}}, \bar{\mathtt{S}}\mathtt{C}, \bar{\mathtt{S}}\bar{\mathtt{C}}\}$. The only non-identity reset occurs when the system transitions from $\mathcal{D}_{\bar{\mathtt{S}}\bar{\mathtt{C}}}$ to $\mathcal{D}_{\bar{\mathtt{S}}\mathtt{C}}$.

and the index set is given by $\mathcal{J} = \{\mathtt{S}\bar{\mathtt{C}}, \bar{\mathtt{S}}\bar{\mathtt{C}}, \bar{\mathtt{S}}\mathtt{C}, \bar{\mathtt{S}}\bar{\mathtt{C}}\}$. Furthermore, $\mathcal{F}_{\mathtt{S}\mathtt{C}} = \mathcal{F}_{\mathtt{S}\bar{\mathtt{C}}} = \mathcal{F}_{\bar{\mathtt{S}}\bar{\mathtt{C}}} = \mathcal{F}_{\text{uncon}}$ and $\mathcal{F}_{\bar{\mathtt{S}}\mathtt{C}} = \mathcal{F}_{\text{con}}$. As a mnemonic, $\mathtt{S}$ indicates that $\Delta_1(x_1) \le S_2(x_2)$ so that adequate downstream supply is available, and $\bar{\mathtt{S}}$ indicates the opposite. Similarly, $\mathtt{C}$ indicates the status of the hysteresis effect so that the congestion mode is only possible for domains with $\mathtt{C}$, and impossible for domains with $\bar{\mathtt{C}}$.

Define the guard functions

$$g_{\mathtt{S}\mathtt{C},\bar{\mathtt{S}}\mathtt{C}}(x) = g_{\mathtt{S}\bar{\mathtt{C}},\bar{\mathtt{S}}\bar{\mathtt{C}}}(x) = S_2(x_2) - \Delta_1(x_1), \tag{84}$$

$$g_{\bar{\mathtt{S}}\mathtt{C},\mathtt{S}\mathtt{C}}(x) = g_{\bar{\mathtt{S}}\bar{\mathtt{C}},\mathtt{S}\bar{\mathtt{C}}}(x) = \Delta_1(x_1) - S_2(x_2), \tag{85}$$

$$g_{\mathtt{S}\bar{\mathtt{C}},\mathtt{S}\mathtt{C}}(x) = g_{\bar{\mathtt{S}}\bar{\mathtt{C}},\bar{\mathtt{S}}\mathtt{C}}(x) = \bar{x}_2 - x_2, \tag{86}$$

$$g_{\mathtt{S}\mathtt{C},\mathtt{S}\bar{\mathtt{C}}}(x) = x_2 - \underline{x}_2. \tag{87}$$

If no guard function is specified between two domains, then no transition is possible between those domains. For all $j, j' \in \mathcal{J}$ such that $g_{j,j'}$ is defined, let $\mathcal{G}_{j,j'} = \{x : g_{j,j'}(x) \le 0\} \cap \mathcal{D}_j$, and let $\mathcal{G}_j = \cup_{j' \in \mathcal{J}} \mathcal{G}_{j,j'}$ for each $j \in \mathcal{J}$.

We have that

$$J_{\text{uncon}}(x) = D_x \mathcal{F}_{\text{uncon}}(x,t) = \begin{bmatrix} -D\Delta_1(x_1) & 0 \\ D\Delta_1'(x_1) & -D\Delta_2(x_2) \end{bmatrix}, \tag{88}$$

$$J_{\text{con}}(x) = D_x \mathcal{F}_{\text{con}}(x,t) = \begin{bmatrix} 0 & -DS_2(x_2) \\ 0 & DS_2(x_2) - D\Delta_2(x_2) \end{bmatrix}. \tag{89}$$

Let $|\cdot|_1$ be the standard one-norm and μ_1 the corresponding matrix measure. It can be verified that

$$\mu_1(J_{\text{uncon}}(x)) \leq 0 \ \forall x \in \mathcal{X}, \quad \text{and} \tag{90}$$
$$\mu_1(J_{\text{con}}(x)) \leq 0 \ \forall x \in \mathcal{X}. \tag{91}$$

Now consider a trajectory in domain $\mathcal{D}_{\bar{\mathsf{S}}\bar{\mathsf{C}}}$ transitioning to $\mathcal{D}_{\bar{\mathsf{S}}\mathsf{C}}$ so that $S_2(x_2) \leq \Delta_1(x_1)$ and the system experiences a capacity drop so that the dynamics transition from uncongested to congested. Computing the saltation matrix Ξ for x such that $g_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(x) = 0$, we have

$$\Xi_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(t,x) = I + \frac{(\mathcal{F}_{\text{con}}(t,x) - \mathcal{F}_{\text{uncon}}(t,x)) \cdot D_x g_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(x)}{D_x g_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(x) \cdot \mathcal{F}_{\text{uncon}}(t,x)} \tag{92}$$
$$= I + \frac{-1}{\Delta_1(x_1) - \Delta_2(\bar{x}_2)} \begin{bmatrix} \Delta_1(x_1) - S_2(\bar{x}_2) \\ S_2(\bar{x}_2) - \Delta_1(x_1) \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \end{bmatrix} \tag{93}$$

for all $x \in \{x : x_2 = \bar{x}_2\} = \mathcal{G}_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(x)$. Let $\rho(x_1) = \frac{\Delta_1(x_1) - S_2(\bar{x}_2)}{\Delta_1(x_1) - \Delta_2(\bar{x}_2)}$ so that

$$\Xi_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(t,x) = \begin{bmatrix} 1 & \rho(x_1) \\ 0 & 1 - \rho(x_1) \end{bmatrix} \tag{94}$$

for all $x \in \{x : x_2 = \bar{x}_2\}$. Because $\bar{x}_2 < x_2^{\text{crit}}$, it holds that $\Delta_2(\bar{x}_2) < S_2(\bar{x}_2)$ and therefore

$$0 \leq \rho(x_1) < 1 \quad \forall x_1 \in \{x_1 : \Delta_1(x_1) \geq S_2(\bar{x}_2)\}. \tag{95}$$

Therefore, $\|\Xi_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}(t,x)\|_1 = 1$ for all $x \in \{x : x_2 = \bar{x}_2\} = \mathcal{G}_{\bar{\mathsf{S}}\bar{\mathsf{C}},\bar{\mathsf{S}}\mathsf{C}}$.

For all $(j,j') \neq (\bar{\mathsf{S}}\bar{\mathsf{C}}, \bar{\mathsf{S}}\mathsf{C})$ such that $\mathcal{G}_{j,j'}$ is nonempty, it can be verified that $\mathcal{F}_{j'}(x) = \mathcal{F}_j(x)$ for all $x \in \mathcal{G}_{j,j'}$ so that $\Xi_{j,j'}(t,x) = I$ and trivially $\|\Xi_{j,j'}(t,x)\|_1 = 1$. Applying Theorem 1, we conclude that

$$|y(t) - x(t)|_1 \leq |y(0) - x(0)|_1 \tag{96}$$

for any pair of trajectories $x(t), y(t)$ of the traffic flow system with initial conditions $y(0), x(0)$ subject to any input $u(t)$, that is, the system is nonexpansive.

In fact, it is possible to conclude that $\lim_{t\to\infty} |x(t) - \gamma(t)|_1 = 0$ for any initial condition $x(0)$ using an approach analogous to that used in [37, Example 4], which considers contraction in traffic flow without modeling capacity drop. In particular, if the derivatives of Δ_1, Δ_2, and S_2 are bounded away from zero, and $u(t)$ is periodic with period T and is such that there exists a periodic orbit $\gamma(t)$ of the hybrid system such that $\gamma(t^*) \in \text{int}(\mathcal{D} \backslash \mathcal{D}_{\bar{\mathsf{S}}\mathsf{C}})$ for some t^*, then the system is strictly contracting towards $\gamma(t)$ for a portion of each period T. This implies that eventually, each trajectory converges to $\gamma(t)$.

B. Mechanical systems subject to unilateral constraints

In this section we consider a subclass[9] of *mechanical* systems with $d \in \mathbb{N}$ degrees–of–freedom (DOF) subject to $n \in \mathbb{N}$ *unilateral* constraints [24],

$$M\ddot{q} = f(t,q,\dot{q}) \text{ s.t. } a(q) \geq 0, \tag{97a}$$
$$\dot{q}^+ = \Delta \dot{q}^-, \ a(q) = 0, \tag{97b}$$

where: $q \in Q = \mathbb{R}^d$ are generalized coordinates; M is the mass matrix; the applied force f consists of an open–loop input $u(t)$ and a linear spring–damper network with stiffness K and damping B, i.e. $f(t,q,\dot{q}) = u(t) - Kq - B\dot{q}$ where $K = K^\top > 0$, $B = B^\top > 0$; the constraint function $a : \mathbb{R}^d \to \mathbb{R}^n$ is linear[10], $a(q) = Da \cdot q$; and we regard the inequality $a(q) \geq 0$ as being enforced componentwise.

[9]To simplify the exposition, we restrict our attention in this section to a subclass of mechanical systems with no Coriolis forces, frictionless linear constraints, and linear spring–damper dynamics. We emphasize that there is no obstacle, in principle, to applying our theoretical results to the general class of mechanical systems subject to unilateral constraints.

[10]If the constraint function is affine, $a(q) = Da \cdot (q - q_0)$, we translate coordinates as $\overline{q} = q - q_0$; this shift offsets the potential energy by an affine term $q^\top K q_0 + \frac{1}{2} q_0^\top K q_0$, so we incorporate the additional constant force generated by this offset, $-Kq_0$, into the open–loop input, $\overline{u}(t) = u(t) - Kq_0$.

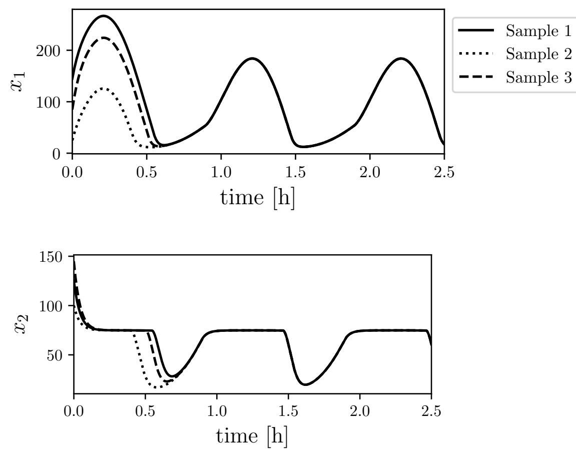


Fig. 5: Sample trajectories of the two link traffic network with capacity drop modeled as hysteresis with a periodic input. All trajectories contract to a unique periodic trajectory.

Given any $J \subset \{1, \ldots, n\}$, and with $\#(J)$ denoting the number of elements in the set J, we let $a_J \colon Q \to \mathbb{R}^{\#(J)}$ denote the function obtained by selecting the component functions of a indexed by J, and say that the constraints in J are *active* at $q \in Q$ if $a_J(q) = 0$. In this case the (perfect, holonomic, scleronomic)[11] constraints apply additional forces $Da_J^\top \lambda_J$ where

$$\lambda_J = -\left(Da_J M^{-1} Da_J^\top\right)^{-1} Da_J M^{-1} f; \tag{98}$$

these forces ensure $a_J(q) \geq 0$ in (97a). When new constraints activate, they apply impulses to prevent constraint violation; we consider perfectly plastic impact in (97b), i.e. we let

$$\Delta_J = I - M^{-1} Da_J^\top \left(Da_J M^{-1} Da_J^\top\right)^{-1} Da_J. \tag{99}$$

Restricting to the case where the input is zero, $u \equiv 0$, note that the sum of potential and kinetic energy decreases monotonically in (97) since (97a) is dissipative and (97b) is an orthogonal projection with respect to the mass matrix. This observation led us to initially intuit that the hybrid system[12] specified by (97) ought to be (infinitesimally) contractive. However, as the sequence of examples below illustrates, the interaction of the dissipative flows and resets generally results in infinitesimal *expansion* when constraints activate. Thus, in the remainder of this section we will assess infinitesimal contractivity in simple variants of these systems with few (1 or 2) degrees–of–freedom (DOF), *hard* or *soft* constraints, and *elastic* or *viscoelastic* spring–dampers, in an effort to highlight intrinsic obstacles to infinitesimal contractivity in (97):

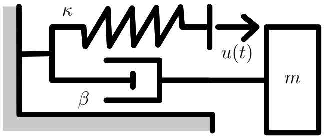
V-B1 presents a minimal example with 1 DOF ($Q = \mathbb{R}^1$) and 1 constraint ($a : Q \to \mathbb{R}^1$), the so–called *linear impact oscillator* [38], finding that continuous–time flow and discrete–time reset are non–expansive;

V-B2 considers a slightly more general system than Section V-B1 with 2 DOF ($Q = \mathbb{R}^2$) and 1 constraint ($a : Q \to \mathbb{R}^1$), finding that continuous–time flow is non–expansive but discrete–time reset is expansive when constraints activate;

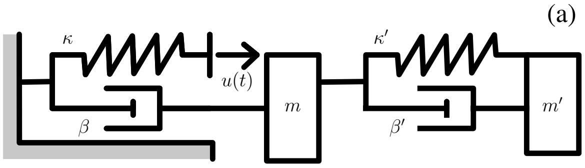
V-B3 relaxes the hard unilateral constraints from Section V-B2 using the penalty method considered in [39], finding that continuous–time flow is non–expansive but discrete–time reset is expansive when constraints *de*activate;

[11] A constraint is: *perfect* if it only generates force in the direction normal to the constraint surface; *holonomic* if it varies with configuration but not velocity; *scleronomic* if it does not vary with time.

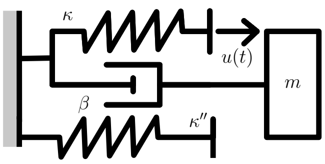
[12] The collection of constraint modes $\{J \subset \{1, \ldots, n\}\}$ with corresponding vector fields and velocity resets together determine a hybrid system in the framework from Section III; we refer the interested reader to [24] for more details.



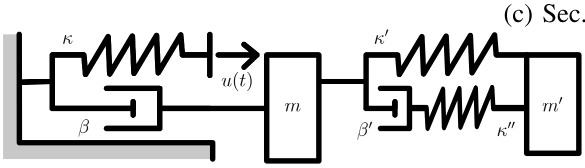
(a) Sec. V-B1: contractive



(b) Sec. V-B2: expansive at impact



(c) Sec. V-B3: expansive at liftoff



(d) Sec. V-B4: contractive

Fig. 6: Mechanical systems subject to unilateral constraints considered in Sec. V-B.

V-B4 modifies the damper in Section V-B2 to obtain a *visco*elastic model, finding that continuous–time flow and discrete–time reset are non–expansive.

These mechanical systems are illustrated in Fig. 6.

1) Elastic spring–damper with 1 *DOF,* 1 *hard constraint:* A point mass m moves along a frictionless rail to the right of a mechanical stop positioned at the origin, impacting plastically if it reaches the stop with negative velocity. The mass is connected to the stop with a parallel spring–damper: the viscosity of the damper is denoted β; the stiffness of the spring is denoted κ; the spring's rest length is adjusted by a time–varying input $u(t)$. This mechanical system subject to a unilateral constraint can be modeled in the hybrid systems framework from Section III with:

- $\mathcal{D}$ $= \mathcal{D}_\emptyset \coprod \mathcal{D}_1$ where $\mathcal{D}_\emptyset = \{(q, \dot{q}) \in [0, +\infty) \times \mathbb{R}\}$ is the set of continuous states wherein the mass is unconstrained and $\mathcal{D}_1 = \{0 \in \mathbb{R}^0\}$ is the (singleton) set of continuous states wherein the mass is constrained;
- $\mathcal{F}$: $[0, \infty) \times \mathcal{D} \to \mathcal{D}$ defined by $\mathcal{F}_\emptyset(t, q, \dot{q}) = (\dot{q}, \ddot{q})$ and $\mathcal{F}_1(t, 0) = 0$, where

$$\ddot{q} = \frac{1}{m}\left(\kappa\left(u(t) - q\right) - \beta\dot{q}\right); \tag{100}$$

- $\mathcal{G}$ $= \mathcal{G}_{\text{TD}} \coprod \mathcal{G}_{\text{LO}}$ where $\mathcal{G}_{\text{TD}} = \{(t, q, \dot{q}) \in [0, \infty) \times \mathcal{D}_\emptyset : q \leq 0,\ \dot{q} < 0\}$ is the set of *touchdown* (TD) states wherein the mass impacts the stop and $\mathcal{G}_{\text{LO}} = \{(t, 0) \in [0, \infty) \times \mathcal{D}_1 : u(t) \geq 0\}$ is the set of *liftoff* (LO) states wherein the mass accelerates off the stop;
- $\mathcal{R}$: $\mathcal{G} \to \mathcal{D}$ defined by $\mathcal{R}|_{\mathcal{G}_{\text{TD}}}(t, q, \dot{q}) = 0 \in \mathbb{R}^0$ (i.e. the position and velocity of the mass are "forgotten" since they are both equal to zero following plastic impact with the stop) and $\mathcal{R}|_{\mathcal{G}_{\text{LO}}}(t, 0) = (0, 0) \in \mathbb{R}^2$ (i.e. the position and velocity of the mass are reinitialized at zero when the mass accelerates off the stop).

Note that it was parsimonious but not necessary to remove the mass's position and velocity from the continuous state in the constrained mode; we elect to remove these states here and in what follows primarily to illustrate application of this paper's theoretical results in systems wherein the dimension of the continuous state varies in different discrete modes.

Letting the sum of the potential and kinetic energy $e(q, \dot{q}) = \frac{1}{2}\kappa(u(t) - q) + \frac{1}{2}m\dot{q}^2$ determine the weighting matrix for a 2–norm[13] in the unconstrained mode,

$$E = D^2 e = \begin{bmatrix} \kappa & 0 \\ 0 & m \end{bmatrix}, \tag{101}$$

and recalling that the matrix measure for a 2–norm determined by weighting matrix E can be computed as

$$\mu(X) = \max \operatorname{spec} \frac{1}{2}\left(X^\top \cdot E + E \cdot X\right), \tag{102}$$

we find that

$$\frac{1}{2}\left(D_x \mathcal{F}_\emptyset^\top \cdot E + E \cdot D_x \mathcal{F}_\emptyset\right) = \begin{bmatrix} 0 & 0 \\ 0 & -\beta \end{bmatrix}, \tag{103}$$

whence the unconstrained flow is non–expansive overall and contractive in velocity. Since $D_x \mathcal{F}_1 = 0$, the constrained flow is non–expansive. The saltation matrices for constraint activation and deactivation are both zero operators, hence their induced norms are zero, whence discrete–time reset is contractive. Theorem 1 implies that the distance between trajectories does not increase over time. Corollary 1 yields the intuitive conclusion that the distance between any two trajectories is zero if both trajectories have undergone at least one discrete transition.

2) Elastic spring–damper with 2 *DOF,* 1 *hard constraint:* Two point masses m, m' move along a frictionless rail to the right of a stop positioned at the origin; mass m impacts plastically if it reaches the stop with negative velocity. Parallel linear spring–dampers connect the masses to one another, and mass m to the stop; the viscosity of the damper connecting m to the stop is denoted β (resp. β' for the damper connecting the two masses); stiffness of the spring connecting m to the stop is denoted by κ (resp. κ'); and the rest length of this spring is adjusted by a time–varying input $u(t)$. This mechanical system subject to a unilateral constraint can be modeled in the hybrid systems framework from Section III with:

$\mathcal{D}$ $= \mathcal{D}_\emptyset \coprod \mathcal{D}_1$ where $\mathcal{D}_\emptyset = \{(q, q', \dot{q}, \dot{q}') \in [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}$ is the set of continuous states wherein mass m is unconstrained and $\mathcal{D}_1 = \{(q', \dot{q}') \in \mathbb{R} \times \mathbb{R}\}$ is the set of continuous states wherein mass m is constrained;

$\mathcal{F}$: $[0, \infty) \times \mathcal{D} \to \mathcal{D}$ defined by $\mathcal{F}_\emptyset(t, q, q', \dot{q}, \dot{q}') = (\dot{q}, \dot{q}', \ddot{q}, \ddot{q}')$ and $\mathcal{F}_1(t, q', \dot{q}') = (\dot{q}', \ddot{q}')$ where

$$\begin{aligned} \ddot{q} &= \frac{1}{m}\left(+\kappa\left(q' - q\right) + \beta\left(\dot{q}' - \dot{q}\right) + \kappa(u(t) - q) - \beta\dot{q}\right), \\ \ddot{q}' &= \frac{1}{m'}\left(-\kappa\left(q' - q\right) - \beta\left(\dot{q}' - \dot{q}\right)\right); \end{aligned} \tag{104}$$

$\mathcal{G}$ $= \mathcal{G}_{\text{TD}} \coprod \mathcal{G}_{\text{LO}}$ where

$$\mathcal{G}_{\text{TD}} = \{(t, q, q', \dot{q}, \dot{q}') \in [0, \infty) \times \mathcal{D}_\emptyset : q \leq 0,\ \dot{q} < 0\} \tag{105}$$

is the set of *touchdown* (TD) states wherein mass m impacts the stop and

$$\mathcal{G}_{\text{LO}} = \{(t, q', \dot{q}') \in [0, \infty) \times \mathcal{D} : \ddot{q} \geq 0\} \tag{106}$$

is the set of *liftoff* (LO) states wherein mass m' accelerates off the stop;

$\mathcal{R}$: $\mathcal{G} \to \mathcal{D}$ defined by $\mathcal{R}|_{\mathcal{G}_{\text{TD}}}(t, q, q', \dot{q}, \dot{q}') = (q', \dot{q}')$ (i.e. the position and velocity of mass m are "forgotten" since they are both equal to zero when the stop constraint activates) and $\mathcal{R}|_{\mathcal{G}_{\text{LO}}}(t, q', \dot{q}') = (0, q', 0, \dot{q}')$ (i.e. the position and velocity of mass m are reinitialized at zero when stop constraint deactivates).

[13]That is, $|x| = \sqrt{\frac{1}{2} x^\top E\, x}$.

As with the previous example, we let the sum of potential and kinetic energy determine the weighting matrices for 2–norms that will be used to assess infinitesimal contractivity of continuous–time flow and discrete–time reset. In the unconstrained mode, the energy is

$$e_\emptyset(q,\dot{q}) = \frac{1}{2}\kappa(u(t)-q)^2 + \frac{1}{2}\kappa'\left(q'-q\right)^2 + \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m'\dot{q}'^2, \tag{107}$$

yielding the metric

$$E_\emptyset = D^2 e_\emptyset = E_\emptyset = \begin{bmatrix} \kappa+\kappa' & -\kappa' & 0 & 0 \\ -\kappa' & \kappa & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m' \end{bmatrix}. \tag{108}$$

In the constrained mode, the energy simplifies to

$$e_1(q,\dot{q}) = \frac{1}{2}\kappa u(t)^2 + \frac{1}{2}\kappa' q'^2 + \frac{1}{2}m'\dot{q}'^2, \tag{109}$$

yielding the metric

$$E_1 = D^2 e_1 = \begin{bmatrix} \kappa' & 0 \\ 0 & m' \end{bmatrix}. \tag{110}$$

We first consider infinitesimal contractivity of continuous–time flow. Letting $x = (q, q', \dot{q}, \dot{q}')$ denote the continuous state vector in the unconstrained mode and $\dot{x} = \mathcal{F}_\emptyset(t,x)$ denote its time derivative yields

$$\frac{1}{2}\left(D_x\mathcal{F}_\emptyset^\top \cdot E_\emptyset + E_\emptyset \cdot D_x\mathcal{F}_\emptyset\right) = \operatorname{diag}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -(\beta+\beta') & \beta' \\ \beta' & -\beta' \end{bmatrix}\right), \tag{111}$$

Similarly, letting $x = (q', \dot{q}')$ denote the continuous state vector in the constrained mode and $\dot{x} = \mathcal{F}_1(t,x)$ denote its time derivative yields

$$\frac{1}{2}\left(D_x\mathcal{F}_1^\top \cdot E_1 + E_1 \cdot D_x\mathcal{F}_1\right) = \operatorname{diag}\left(0, -\beta'\right). \tag{112}$$

The spectrum of

$$\frac{1}{2}\left(D_x\mathcal{F}_\emptyset^\top \cdot E_\emptyset + E_\emptyset \cdot D_x\mathcal{F}_\emptyset\right) \tag{113}$$

is $\left\{0, -\frac{1}{2}\left(\beta + 2\beta' \pm \sqrt{\beta^2 + 4\beta'^2}\right)\right\}$ and that of

$$\frac{1}{2}\left(D_x\mathcal{F}_1^\top \cdot E_1 + E_1 \cdot D_x\mathcal{F}_1\right) \tag{114}$$

is $\{0, -\beta'\}$, so the matrix measures of $D_x\mathcal{F}_\emptyset$ and $D_x\mathcal{F}_1$ are both equal to 0 (zero), whence continuous–time flow is non–expansive[14] in both the constrained and unconstrained mode.

We now consider infinitesimal contractivity of discrete–time reset, i.e. we evaluate induced norms of saltation matrices. The *liftoff* (LO) saltation matrix is

$$\Xi_{\text{LO}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \tag{115}$$

since this matrix is an energy–preserving embedding, it is an isometry with respect to the 2–norms determined by the energy metric, hence its induced norm is equal to unity. The *touchdown* (TD) saltation matrix is

$$\Xi_{\text{TD}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{m'}\beta' & 0 & 0 & 1 \end{bmatrix}.$$

[14]Since some of the eigenvalues are negative, the flow is actually *semi*– [1, Sec. 2] or *horizontally* [40, Sec. VII] contractive.

We seek to evaluate the matrix norm $|\Xi_{TD}|$ induced by the vector 2–norms determined by $E_\emptyset$ in the unconstrained mode and E_1 in the constrained mode:

$$|\Xi_{\text{TD}}| = \max\left\{|\Xi_{\text{TD}}x|_1 : |x|_\emptyset = 1\right\}.$$

Unfortunately, although an expression for this induced norm is readily obtained using symbolic computer algebra, we were unable to analytically determine when this expression is larger than unity. However, a straightforward calculation shows that the induced norm is larger than unity for all β' sufficiently large: noting that the vector $v = \left(\frac{\sqrt{2}}{\sqrt{\kappa+\kappa'}}, 0, 0, 0\right)^\top \in \mathcal{D}_\emptyset$ has norm $|v|_\emptyset = 1$ and that the vector $w = \Xi_{\text{TD}}v = \left(0, -\frac{\sqrt{2}}{m'\sqrt{\kappa+\kappa'}}\beta'\right)^\top \in \mathcal{D}_1$ has norm $|w|_1 = \frac{1}{\sqrt{m'(\kappa+\kappa')}}\beta'$, we conclude that $|\Xi_{\text{TD}}| \geq |w|_1$ and hence $|\Xi_{\text{TD}}| > 1$ for all β' sufficiently large. (Numerical experiments[15] indicate that the induced norm is larger than unity for all $\beta' > 0$.) We conclude that constraint activation is generally expansive.

3) Elastic spring–damper with soft constraints: The result in the previous section indicates that spring–damper networks subject to *hard* unilateral constraints generally do not satisfy the discrete–time infinitesimal contractivity condition (35) in Theorem 1 when the system has more than a single degree–of–freedom (the 1–DOF example from Sec. V-B1 is contractive only because the constrained mode is zero–dimensional). The reset, or restitution law, used to model impacts against hard constraints coarsely approximates the actual mechanics of the interaction between bodies, which consist of elastic and plastic deformation in the contact zone. An alternative approach is to explicitly model this deformation using additional forces. In this section, we consider infinitesimal contractivity of the class of mechanical systems subject to *soft* unilateral constraints studied in [39]. Specifically, rather than exactly enforcing unilateral constraints $a(q) \geq 0$ in (97), we will *penalize* constraint violation using a potential function that applies forces "as though a linear elastic spring were located at the point of contact" [39, Sec II-B], yielding the modified potential energy

$$v_J(q) = \frac{1}{2}q^\top K q + \frac{1}{2}\sum_{j\in J} k_j a_j(q)^2 = \frac{1}{2}q^\top \left(K + K_J\right) q \tag{116}$$

where $K_J = Da_J^\top \operatorname{diag}\left\{k_j\right\}_{j\in J} Da_J$ is a positive semidefinite stiffness matrix for the potential v_J associated with the subset of constraints $J \subset \{1, \dots, n\}$ that are active, i.e. for which $a_J(q) \leq 0$. With this modification, the system's equations of motion become

$$M\ddot{q} = u(t) - (K + K_J)q - B\dot{q},\ \ a_J(q) \leq 0. \tag{117}$$

The dynamics in (117) are classical in the sense that the right–hand side of the equation specifies a Lipschitz continuous and piecewise–differentiable vector field. However, the natural distance metric determined by the sum of potential and kinetic energy,

$$e_J(q, \dot{q}) = \frac{1}{2}q^\top \left(K + K_J\right) q + \frac{1}{2}\dot{q}^\top M \dot{q}, \tag{118}$$

now depends on the set of active constraints J, whence the weighting matrix for the energy–induced 2–norm also depends on the set of active constraints,

$$E_J = D^2 e_J = \begin{bmatrix} K + K_J & 0 \\ 0 & M \end{bmatrix}. \tag{119}$$

Letting $x = (q, \dot{q})$ and $\dot{x} = \mathcal{F}_J(t, x)$ for $a_J(q) \leq 0$, using the energy–induced 2–norm yields a negative semidefinite matrix

$$\frac{1}{2}\left(D_x\mathcal{F}_J^\top \cdot E_J + E_J \cdot D_x\mathcal{F}_J\right) = \begin{bmatrix} 0 & 0 \\ 0 & -B \end{bmatrix}, \tag{120}$$

[15] We sampled $m' \in (0, 10)$, $\kappa \in (0, 1000)$, $\kappa' \in (0, 1000)$, $\beta' \in (0, 10)$ uniformly at random 100 000 times.

whereas using, e.g., the 2–norm from the unconstrained mode yields

$$\begin{aligned} \left[\begin{array}{cc} q^\top & \dot{q}^\top \end{array}\right] \left(D_x\mathcal{F}_J^\top \cdot E_\emptyset + E_\emptyset \cdot D_x\mathcal{F}_J\right) \left[\begin{array}{c} q \\ \dot{q} \end{array}\right] &= \left[\begin{array}{cc} q^\top & \dot{q}^\top \end{array}\right] \left[\begin{array}{cc} 0 & -\frac{1}{2}K_J \\ -\frac{1}{2}K_J & -B \end{array}\right] \left[\begin{array}{c} q \\ \dot{q} \end{array}\right] \\ &= -\dot{q}^\top K_J q - 2\dot{q}^\top B\dot{q}, \end{aligned} \tag{121}$$

which is not a negative semidefinite quadratic form in any constrained mode (i.e. when $J \neq \emptyset$). In the remainder of this section we will assess infinitesimal contractivity of (117) using the energy–induced norms that depend on the set of active constraints.

Infinitesimal contractivity for continuous and piecewise–differentiable vector fields where different norms (hence, matrix measures) are associated with each differentiable "piece" of the vector field have previously been considered in [17], but only for *switched* systems where the discrete transition between "pieces" is triggered by an exogenous input, i.e. does not depend on the continuous state. As the previous sections illustrate, the interaction between continuous–time (97a) and discrete–time (97b) dynamics can yield expansion even when the continuous and discrete components are individually non–expansive. We will assess infinitesimal contractivity of this system by treating (117) as a hybrid system and applying our results; for ease of exposition, we will restrict our attention to the system from Section V-B1 with 1 DOF and 1 constraint.

Noting that positions, velocities, and accelerations are continuous in (117) when constraints (de)activate, we conclude that both saltation matrices are the 2–dimensional identity, $\Xi_{\emptyset,1} = \Xi_{1,\emptyset} = I_2$. Thus, the induced norms can be computed as $|\Xi_{\emptyset,1}| = \sigma_{\max}\left(S_1\Xi_{\emptyset,1}S_\emptyset^{-1}\right)$, $|\Xi_{1,\emptyset}| = \sigma_{\max}\left(S_1\Xi_{\emptyset,1}S_\emptyset^{-1}\right)$ with

$$S_\emptyset = \left[\begin{array}{cc} \sqrt{\kappa} & 0 \\ 0 & \sqrt{m} \end{array}\right],\ S_1 = \left[\begin{array}{cc} \sqrt{\kappa+\kappa''} & 0 \\ 0 & \sqrt{m} \end{array}\right] \tag{122}$$

denoting the square roots of $E_\emptyset$, E_1, respectively, yielding

$$\begin{aligned} |\Xi_{\emptyset,1}| &= \sigma_{\max}\left(S_1 S_\emptyset^{-1}\right) = \sqrt{\frac{\kappa}{\kappa+\kappa''}} < 1, \\ |\Xi_{1,\emptyset}| &= \sigma_{\max}\left(S_\emptyset S_1^{-1}\right) = |\Xi_{\emptyset,1}|^{-1} > 1. \end{aligned} \tag{123}$$

We conclude constraint activation is contractive and constraint deactivation is expansive.

4) Viscoelastic spring–damper with 2 *DOF,* 1 *hard constraint:* We now return to the example from Section V-B2, wherein two point masses m, m' move along a frictionless rail to the right of a stop positioned at the origin and mass m impacts plastically if it reaches the stop with negative velocity. In that section, we found that discrete–time reset was generally expansive during constraint activation due to the discontinuous change in force produced by the damper that connects m and m'. Since this discontinuous change in force was caused by the discontinuous change in the velocity of m during constraint activation, we considered the effect of *softening* the constraint in Section V-B3; although this approach led to non–expansion during constraint activation as expected, it unexpectedly introduced expansion during constraint deactivation.

Since the structures connecting bodies in real mechanical systems allow deformation (e.g. the elastic bending of a robot joint), discontinuous force production is physically implausible—the additional deflections smooth forces transmitted between elements. In an effort to obtain the non–expansive behavior we expect from systems like (97), we now consider a different variation of the model with rigid constraints wherein the viscous damper connecting the two masses is replaced with a *visco*elastic element. Specifically, the damper connecting the two masses is replaced by a series spring–damper with the same viscosity β' and new spring stiffness κ'' as illustrated in Figure 6.

This mechanical system subject to a unilateral constraint can be modeled in the hybrid systems framework from Section III with:

- $\mathcal{D} = \mathcal{D}_\emptyset \coprod \mathcal{D}_1$ where $\mathcal{D}_\emptyset = \{(q, q', \ell, \dot{q}, \dot{q}') \in [0,\infty) \times \mathbb{R}^4\}$ is the set of continuous states wherein mass m is unconstrained and $\mathcal{D}_1 = \{(q', \ell, \dot{q}') \in \mathbb{R}^3\}$ is the set of continuous states wherein mass m is constrained;

$\mathcal{F}$: $[0,\infty)\times\mathcal{D}\to\mathcal{D}$ defined by $\mathcal{F}_\emptyset(t,q,q',\ell,\dot q,\dot q') = (\dot q,\dot q',\dot\ell,\ddot q,\ddot q')$ and $\mathcal{F}_1(t,q',\ell,\dot q') = (\dot q',\dot\ell,\ddot q')$ where

$$\begin{aligned}\dot\ell &= \frac{\kappa''}{\beta'}\,(q'-q-\ell)\\ \ddot q &= \frac{1}{m}\,(+\kappa''(q'-q-\ell)+\kappa'\,(q'-q)\\ &\qquad + \kappa\,(u-q)-\beta\dot q),\\ \ddot q' &= \frac{1}{m'}\,(-\kappa''\,(q'-q-\ell)-\kappa'\,(q'-q))\end{aligned}$$

$\mathcal{G}$ $=\mathcal{G}_{\text{TD}}\coprod\mathcal{G}_{\text{LO}}$ where

$$\mathcal{G}_{\text{TD}} = \{(t,q,q',\ell,\dot q,\dot q')\in[0,\infty)\times\mathcal{D}_\emptyset : q\le 0,\ \dot q<0\} \tag{124}$$

is the set of *touchdown* (TD) states wherein mass m impacts the stop and

$$\mathcal{G}_{\text{LO}} = \{(t,q',\ell,\dot q')\in[0,\infty)\times\mathcal{D} : \ddot q\ge 0\} \tag{125}$$

is the set of *liftoff* (LO) states wherein mass m' accelerates off the stop;

$\mathcal{R}$: $\mathcal{G}\to\mathcal{D}$ defined by $\mathcal{R}|_{\mathcal{G}_{\text{TD}}}(t,q,\dot q,q',\dot q') = (q',\dot q')$ (i.e. the position and velocity of mass m are "forgotten" since they are both equal to zero when the stop constraint activates) and $\mathcal{R}|_{\mathcal{G}_{\text{LO}}}(t,q',\dot q') = (0,q',0,\dot q')$ (i.e. the position and velocity of mass m are reinitialized at zero when stop constraint deactivates).

As with the previous example, we let the sum of potential and kinetic energy determine the weighting matrices for 2–norms that will be used to assess infinitesimal contractivity of continuous–time flow and discrete–time reset. In the unconstrained mode, the energy is

$$\begin{aligned}e_\emptyset(q,\dot q) &= \frac{\kappa}{2}\,(u(t)-q)^2+\frac{\kappa'}{2}\,(q'-q)^2+\frac{\kappa''}{2}\,(q'-q-\ell)^2\\ &\quad+\frac{\dot q^2 m}{2}+\frac{\dot q'^2 m'}{2}\end{aligned}$$

yielding the metric

$$E_\emptyset = D^2 e_\emptyset = \begin{bmatrix}\kappa+\kappa'+\kappa'' & -\kappa'-\kappa'' & \kappa'' & 0 & 0\\ -\kappa'-\kappa'' & \kappa'+\kappa'' & -\kappa'' & 0 & 0\\ \kappa'' & -\kappa'' & \kappa'' & 0 & 0\\ 0 & 0 & 0 & m & 0\\ 0 & 0 & 0 & 0 & m'\end{bmatrix}. \tag{126}$$

In the constrained mode, the energy simplifies to

$$e_1(q,\dot q) = \frac{\kappa u(t)^2}{2}+\frac{\kappa' q'^2}{2}+\frac{\kappa''}{2}\,(q'-\ell)^2+\frac{\dot q'^2 m'}{2} \tag{127}$$

yielding the metric

$$E_1 = D^2 e_1 = \begin{bmatrix}\kappa'+\kappa'' & -\kappa'' & 0\\ -\kappa'' & \kappa'' & 0\\ 0 & 0 & m'\end{bmatrix}. \tag{128}$$

We first consider infinitesimal contractivity of continuous–time flow. Letting $x=(q,q',\dot q,\dot q')$ denote the continuous state vector in the unconstrained mode and $\dot x=\mathcal{F}_\emptyset(t,x)$ denote its time derivative yields

$$\frac{1}{2}\left(D_x\mathcal{F}_\emptyset^\top\cdot E_\emptyset + E_\emptyset\cdot D_x\mathcal{F}_\emptyset\right) = \operatorname{diag}\left(\begin{bmatrix}-\frac{\kappa''^2}{\beta'} & \frac{\kappa''^2}{\beta'} & -\frac{\kappa''^2}{\beta'}\\ \frac{\kappa''^2}{\beta'} & -\frac{\kappa''^2}{\beta'} & \frac{\kappa''^2}{\beta'}\\ -\frac{\kappa''^2}{\beta'} & \frac{\kappa''^2}{\beta'} & -\frac{\kappa''^2}{\beta'}\end{bmatrix}, \begin{bmatrix}-\beta & 0\\ 0 & 0\end{bmatrix}\right). \tag{129}$$

Similarly, letting $x = (q', \dot{q}')$ denote the continuous state vector in the constrained mode and $\dot{x} = \mathcal{F}_1(t, x)$ denote its time derivative yields

$$\frac{1}{2}\left(D_x\mathcal{F}_1^\top \cdot E_1 + E_1 \cdot D_x\mathcal{F}_1\right) = \operatorname{diag}\left(\begin{bmatrix} -\frac{\kappa''^2}{\beta'} & \frac{\kappa''^2}{\beta'} \\ \frac{\kappa''^2}{\beta'} & -\frac{\kappa''^2}{\beta'} \end{bmatrix}, 0\right). \tag{130}$$

The spectrum of $\frac{1}{2}\left(D_x\mathcal{F}_\emptyset^\top \cdot E_\emptyset + E_\emptyset \cdot D_x\mathcal{F}_\emptyset\right)$ is $\left\{0, -\beta, -\frac{3\kappa''^2}{\beta'}\right\}$ and that of $\frac{1}{2}\left(D_x\mathcal{F}_1^\top \cdot E_1 + E_1 \cdot D_x\mathcal{F}_1\right)$ is $\left\{0, -\frac{2\kappa''^2}{\beta'}\right\}$, so the matrix measures of $D_x\mathcal{F}_\emptyset$ and $D_x\mathcal{F}_1$ are both equal to 0 (zero), whence continuous–time flow is non–expansive[16] in both the constrained and unconstrained mode.

We now consider infinitesimal contractivity of discrete–time reset, i.e. we evaluate induced norms of saltation matrices. The *liftoff* (LO) saltation matrix is

$$\Xi_{\text{LO}} = D_x\mathcal{R}|_{\mathcal{G}_{\text{LO}}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{131}$$

since this matrix is an energy–preserving embedding, it is an isometry with respect to the 2–norms determined by the energy metric, hence its induced norm is equal to unity. Similarly, the *touchdown* (TD) saltation matrix,

$$\Xi_{\text{TD}} = D_x\mathcal{R}|_{\mathcal{G}_{\text{TD}}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{132}$$

is an orthogonal projection with respect to the 2–norms determined by the energy metric, hence its induced norm is equal to unity.

Combining our analyses of infinitesimal contractivity of continuous–time flow and discrete–time reset from the two preceding paragraphs, we conclude that Theorem 1 applies to this system with $c = 0$ in (34) and $K = 1$ in (35), i.e. the system's dynamics are non–expansive.

VI. Converse result

We now present a converse result indicating that the continuous–time (34) and discrete–time (35) contraction conditions of Theorem 1 are tight.

Theorem 2. *Consider a hybrid system satisfying Assumptions 1 and 2 and further assume (a) no guards overlap so that* $\overline{\mathcal{G}}_{j,j'} \cap \overline{\mathcal{G}}_{j,j''} = \emptyset$ *for all* $j' \neq j''$ *and (b) no discrete mode is ever entirely guard so that* $\mathcal{D}_j \backslash \mathcal{G}_j(t) \neq \emptyset$ *for all* j *and all* $t \geq 0$. *If the hybrid system is contractive, i.e., if there exists a constant* $c \in \mathbb{R}$ *such that*[17]

$$d_t(\phi(t, s, \xi), \phi(t, s, \zeta)) \leq e^{c(t-s)} d_s(\xi, \zeta) \tag{133}$$

for all $\xi, \zeta \in \mathcal{D}$, $t \geq s \geq 0$, *then the continuous–time* (34) *and discrete–time* (35) *contraction conditions in Theorem 1 are satisfied:*

$$\mu_j\left(D_x\mathcal{F}_j(t, x)\right) \leq c \tag{134}$$

for all $t \geq 0$ *and all* $x \in \mathcal{D} \backslash \mathcal{G}(t)$, *and*

$$\|\Xi(t, x)\|_{j,j'} \leq 1 \tag{135}$$

[16]As was the case with the example in Section V-B2, the flow is actually *semi*– [1, Sec. 2] or *horizontally* [40, Sec. VII] contractive.

[17]$d_t : \mathcal{M}_t \times \mathcal{M}_t \to [0, \infty)$ is the (time–varying) intrinsic distance metric defined in Sec. III-3 on the hybrid system's quotient space $\mathcal{M}_t$.

for all $j, j' \in \mathcal{J}$, all $t > 0$, and all $x \in \mathcal{G}_{j,j'}(t)$.

Proof. Fix a time $\sigma \geq 0$ and consider $x \in \mathcal{D}_j \backslash \mathcal{G}(\sigma)$ for some $j \in \mathcal{J}$, and recall that $\mathcal{D}_j \backslash \mathcal{G}(\sigma)$ is open. Let $\delta > 0$ be such that $\mathcal{B}_\delta = \{\zeta : |\zeta - x|_j < \delta\} \subset \mathcal{D}_j \backslash \mathcal{G}(\sigma)$, and notice that $d_\sigma(x, \zeta) = |x - \zeta|_j$ for all $\zeta \in \mathcal{B}_\delta$. By standard converse results for continuously differentiable contracting systems, *e.g.*, [6, Proposition 3], $\mu_j\left(D_x \mathcal{F}_j(\sigma, x)\right) \leq c$. Since σ, x, and j where arbitrary, (134) holds.

Now fix a time $\sigma > 0$. Then, from (133), for any $\xi, \zeta \in \mathcal{D}$ and all $\epsilon \in (0, \sigma]$,

$$\frac{d_{\sigma+\epsilon}(\phi(\sigma+\epsilon, \sigma-\epsilon, \xi), \phi(\sigma+\epsilon, \sigma-\epsilon, \zeta))}{d_{\sigma-\epsilon}(\xi, \zeta)} \leq e^{2c\epsilon} \tag{136}$$

whenever $d_{\sigma-\epsilon}(\xi, \zeta) \neq 0$. It follows that

$$\limsup_{\epsilon \to 0^+} \sup_{\xi, \zeta \in \mathcal{D}} \frac{d_{\sigma+\epsilon}(\phi(\sigma+\epsilon, \sigma-\epsilon, \xi), \phi(\sigma+\epsilon, \sigma-\epsilon, \zeta))}{d_{\sigma-\epsilon}(\xi, \zeta)} \leq \lim_{\epsilon \to 0^+} e^{2c\epsilon} = 1. \tag{137}$$

Consider $x^* \in \mathcal{G}_{j,j'}(\sigma)$ for some $j, j' \in \mathcal{D}$. By Assumption 1.4, $D_t g_{j,j'}(\sigma, x^*) + D_x g_{j,j'}(\sigma, x^*) \cdot \mathcal{F}_j(\sigma, x^*) < 0$. Since also $\mathcal{D}_j \backslash \mathcal{G}_j(\sigma) \neq \emptyset$, there exists $x_0 \in \mathcal{D}_j$ and time $\tau < \sigma$ satisfying $\phi_j(\sigma^-, \tau, x_0) = x^*$. Further, because guards do not overlap, there exists an open neighborhood $\mathcal{O} \ni x_0$ such that for all $\xi \in \mathcal{O}$, there exists $\nu(\xi) > \tau$ such that $\phi(\nu(\xi)^-, \tau, \xi) \in \mathcal{G}_{j,j'}(\nu(\xi))$ and $\phi(t, \tau, \xi) \in D_j \backslash \mathcal{G}(t)$ for all $t \in [\tau, \nu(\xi))$. We write $x(t) = \phi(t, \tau, x_0)$ so that, in particular, $x(\tau) = x_0$ and $x(\sigma^-) = x^*$.

Assume the neighborhood $\mathcal{O}$ is chosen small enough so that there exists $\bar{\tau} > 0$ such that $\phi(t, \tau, \xi) \in \mathcal{D}_{j'}$ for all $t \in [\nu(\xi), \sigma + \bar{\tau})$ for all $\xi \in \mathcal{O}$; existence of such a $\bar{\tau}$ for small enough $\mathcal{O}$ is guaranteed by Assumption 1.2. For each $\epsilon \in (0, \min\{\sigma - \tau, \bar{\tau} - \sigma\})$, let $\delta(\epsilon) > 0$ be small enough so that, for all $\xi \in \mathcal{B}_{\delta(\epsilon)}(\sigma - \epsilon, x(\sigma - \epsilon)) = \{\xi \in \mathcal{D} : d_{\sigma-\epsilon}(\xi, x(\sigma - \epsilon)) < \delta(\epsilon)\}$, it holds that

$$d_{\sigma-\epsilon}(\xi, z(\sigma-\epsilon)) = |\xi - z(\sigma-\epsilon)|_j, \tag{138}$$

$$d_{\sigma+\epsilon}(\phi(\sigma+\epsilon, \sigma-\epsilon, \xi), \phi(\sigma+\epsilon, \sigma-\epsilon, z(\sigma-\epsilon))) \tag{139}$$

$$= |\phi(\sigma+\epsilon, \sigma-\epsilon, \xi) - \phi(\sigma+\epsilon, \sigma-\epsilon, z(\sigma-\epsilon))|_{j'}. \tag{140}$$

Then

$$1 \geq \limsup_{\epsilon \to 0^+} \sup_{\xi \in \mathcal{B}_{\delta(\epsilon)}(x(\sigma-\epsilon))} \frac{|\phi(\sigma+\epsilon, \sigma-\epsilon, \xi) - \phi(\sigma+\epsilon, \sigma-\epsilon, x(\sigma-\epsilon))|_{j'}}{|\xi - x(\sigma-\epsilon)|_j} \tag{141}$$

$$\geq \limsup_{\epsilon \to 0^+} \sup_{\xi \in \mathbb{R}^{n_j}} \frac{|D_3\phi(\sigma+\epsilon, \sigma-\epsilon, x(\sigma-\epsilon)) \cdot (\xi - x(\sigma-\epsilon))|_{j'}}{|\xi - x(\sigma-\epsilon)|_j} \tag{142}$$

$$= |\Xi(\sigma, x^*)|_{j,j'}, \tag{143}$$

where the first inequality follows by (137) and the second inequality follows by the definition of directional derivative, completing the proof.

□

Remark 10 (summary of converse result). By restricting our analysis to hybrid systems whose guards are codimension–1 submanifolds in the above Theorem, we found that contraction with respect to the intrinsic distance metric defined in Section III-3 implies infinitesimal contraction in continuous (34) and discrete (35) time. Extensions of this result to more general guard structures are discussed in Section VII-E.

VII. Discussion

Before concluding, we discuss possible extensions and applications of the preceding results and examples.

A. Assumptions 1 and 2

We acknowledge that the Assumptions imposed in Section III are not obviously satisfied in all applications of interest. However, as discussed in the sequence of Remarks that followed the Assumptions, each condition plays a crucial role in the Proof of Theorem 1; if any one condition is violated, then there exist systems that cannot be shown contractive using our approach. Furthermore, recent results provide a broad class of systems that satisfy these Assumptions. For instance, [41] considers a class of discontinuous vector fields that yield continuous and piecewise–differentiable flows. Hybrid systems obtained from such vector fields satisfy Assumption 1 by construction [41, Def. 2] and Assumption 2 by piecewise–differentiability of the flow [41, Thm. 5]; these observations justify application of our results in Section V-A. Related work [42] established conditions under which the hybrid system model of a mechanical system subject to unilateral constraints has a continuous and piecewise–differentiable flow. The conditions on the hybrid system [42, Assumps. 1–4] and properties of its flow [42, Thm. 1] ensure that Assumptions 1 and 2 are satisfied for this class of systems, justifying application of our results to the examples in Section V-B. Broadening the class of systems that are known to satisfy these Assumptions is the subject of ongoing work.

B. Periodic orbits

In some applications [43], the existence of a periodic orbit can be established (or assumed) *a priori*. With our Assumptions 1 and 2 in effect, the orbit's compactness implies there exist finite upper and lower bounds on dwell time for nearby trajectories. These dwell time bounds can, in principle, be employed in Corollary 1 to establish a system's contractivity even in the presence of expansion in continuous–time (51) or discrete–time (52) as in (54).

C. Generalizations of infinitesimal contraction

This paper focused on the strongest possible notion of contraction, requiring that the distance between trajectories decrease in all directions at every instant. Recent work has considered relaxations of these strict requirements, indicating possible routes to extend our approach. Importantly, although we had to introduce new and non–trivial analysis techniques to generalize infinitesimal contraction to hybrid systems (specifically, the intrinsic distance metric and quotient space from Section III-3), the core of our approach leverages the same intuition utilized in classical systems (i.e. differential or difference equations, exclusive "or"): a system is contractive if path lengths decrease in time. We believe this close parallel will facilitate generalization of a variety of classical techniques to the hybrid setting.

D. Linear spring–damper networks with d DOF, n constraints

The analysis in Section V-B admits a straightforward generalization to linear spring–damper networks with arbitrary numbers of degrees–of–freedom (d) and constraints (n). Specifically, it can be shown that continuous–time flow is generally non–expansive overall and contractive in velocity coordinates (i.e. *semi–* [1, Sec. 2] or *horizontally* [40, Sec. VII] contractive), and that discrete–time reset is generally: expansive when constraints activate in elastic systems with rigid constraints (as in Section V-B2); expansive when constraints *de*activate in elastic systems with *soft* constraints (as in Section V-B3); non–expansive in *visco*elastic systems with *rigid* constraints (as in Section V-B4). A detailed exposition and application of this generalization will be the subject of a future publication.

E. Extending the converse result

In Theorem 2, by restricting our attention to infinitesimal contraction through a codimension–1 guard, we found that the continuous–time (34) and discrete–time (35) contraction conditions of Theorem 1 are tight. In systems with overlapping guards so that $\overline{\mathcal{G}}_{j,j'} \cap \overline{\mathcal{G}}_{j,j''} \neq \emptyset$ for some $j' \neq j''$, this converse result applies on any codimension–1 portions of $\mathcal{G}_{j,j'}$ and $\mathcal{G}_{j,j''}$; continuous differentiability of the vector field, guard, and reset ensures the conclusions of the converse result extend to the closure of these codimension–1 sets. If a guard $\mathcal{G}_{j,j'''}$ is contained entirely within a codimension–k set where $k > 1$, then the proof of Theorem 2 can be adapted to establish contraction conditions on the vector field derivative and saltation matrix operator in directions tangent to $\mathcal{G}_{j,j'''}$.

VIII. Conclusion

We generalized infinitesimal contraction analysis to hybrid systems by leveraging local properties of continuous–time flow and discrete–time reset to bound the time rate of change of the intrinsic distance between trajectories. Conversely, we showed that contraction with respect to this intrinsic distance metric implies infinitesimal contraction in continuous and discrete time. In addition to expanding the toolkit for stability analysis in hybrid systems, we provide novel bounds for the intrinsic distance metric even when the system is not contractive.

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