Mixed Monotonicity for Reachability and Safety in Dynamical Systems

Samuel Coogan

Abstract— **A dynamical system is mixed monotone if its vector field or update-map is decomposable into an increasing component and a decreasing component. In this tutorial paper, we study both continuous-time and discrete-time mixed monotonicity and consider systems subject to an input that accommodates, *e.g.*, unknown parameters, an unknown disturbance input, or an exogenous control input. We first define mixed monotonicity with respect to a decomposition function, and we recall sufficient conditions for mixed monotonicity based on sign properties of the state and input Jacobian matrices for the system dynamics. The decomposition function allows for constructing an embedding system that lifts the dynamics to another dynamical system with twice as many states but where the dynamics are monotone with respect to a particular southeast order. This enables applying the powerful theory of monotone systems to the embedding system in order to conclude properties of the original system. In particular, a single trajectory of the embedding system provides hyperrectangular over-approximations of reachable sets for the original dynamics. In this way, mixed monotonicity enables efficient reachable set approximation for applications such as optimization-based control and abstraction-based formal methods in control systems.**

I. Introduction

Reachability analysis of a dynamical system consists of identifying the set of possible future (or past) states given a set of initial states. Reachability has a long and rich history, and methods exist for computing reachable sets for continuous-time systems, discrete-time systems, and hybrid systems, as well as systems with control and/or disturbance inputs. The common message in these approaches is that, in general, they suffer from the curse of dimensionality: obtaining exact or close approximations of reachable sets is often not possible for systems with more than a few states and generally requires significant computational resources.

Yet there is a growing need for fast reachability methods in modern control systems. For example, it is increasingly common to formulate feedback controllers obtained as the solution to an optimization problem that is solved online, perhaps on the order of milliseconds per solution. In this context, the future state of the system often appears as either soft or hard constraints in the optimization problem, and thus efficient reachability methods are critical. As another example, reachability analysis is included as a subroutine in many abstraction-based approaches to formal control verification and synthesis that must be solved for each abstract state of the system [1], [2]; for examples of even modest size, this can quickly amount to thousands or even millions of reachability computations.

In order to address this growing need, the trade-off that has emerged is to exchange accuracy for efficiency. Rather than aiming to compute exact or nearly exact reachable sets, it is often sufficient to obtain merely approximate reachable sets with appropriate guarantees; in applications that consider system safety, a guaranteed over-approximation of the reachable set is generally required. A partial list of methods that aim to efficiently compute over-approximations of reachable sets include ellipsoidal techniques [3], zonotope methods [4], interval analysis [5], [6], eigenvalue perturbation bounds [7], and matrix measures [8], [9].

In this tutorial paper, we consider mixed monotone systems theory for efficient computation of reachable sets and invariant sets for a broad class of nonlinear systems. A dynamical system is mixed monotone if its vector field or update-map is decomposable into an increasing component and a decreasing component. More formally, the system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ is mixed monotone with respect to the decomposition function $d(x, \widehat{x})$ if $f(x) = d(x, x)$ for all x and each d_i is increasing in each x_j for $j \neq i$ and decreasing in each $\widehat{x}_j$ for all j where d_i is the i-th component of d. In the particular case when it is possible to take $d(x, \widehat{x}) = f(x)$, we recover the definition of monotonicity with respect to the standard component-wise partial order on $\mathbb{R}^n$; see [10], [11], [12] for further background on monotone systems. The definitions and characterizations extend to discrete-time systems and systems with disturbance inputs.

Given a mixed monotone system, it is possible to embed its dynamics into a $2n$ dimensional embedding system constructed from the decomposition function. Even if the original system is subject to a disturbance input, the embedding system has no inputs and accommodates the worst-case disturbance. The primary motivation for appealing to mixed monotonicity for reachability is that a single trajectory in the embedding system provides hyperrectangular over-approximations of reachable sets of the original system: the first n-coordinates of the system state in the embedding space provide the lower corner of the hyperrectangle, and the last n-coordinates provide the upper corner. Moreover, an equilibrium in the embedding system implies a robustly forward invariant hyperrectangular set for the original system.

Under mild conditions, every system is mixed monotone with respect to some decomposition function and the proof is constructive, however, the constructed decomposition generally does not have a closed form representation and is not easy to evaluate. Thus, in practice, decomposition functions are often constructed from domain knowledge. In addition,

S. Coogan is with the School of Electrical and Computer Engineering and the School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, 30332, USA sam.coogan@gatech.edu. This work was supported in part by the National Science Foundation under grant ECCS-1749357 and the Air Force Office of Scientific Research under grant FA9550-19-1-0015.

there are special cases for which decomposition functions can be readily constructed if the Jacobian matrix for the system dynamics is appropriately bounded. Moreover, the quality of reachable set approximations is inherently tied to the choice of decomposition function. We demonstrate these fundamental results on several examples and case studies.

The idea of decomposing a dynamical system so that it may be embedded in a higher dimensional monotone system is old; see, e.g., [13] for a historical survey and also [14], [15], [16]. Some of the earliest results that apply this idea for bounding reachable sets and for establishing stability of equilibria are presented in [14], [17]. The foundational paper [15] applies these early results and uses an embedding system to establish stability conditions for certain classes of dynamical systems viewed as the feedback interconnection of monotone systems, an idea further extended in [18]. Together, [13], [15] establish the basis for mixed monotonicity in discrete-time and continuous-time in the form adopted in this tutorial. The paper [19] considers discrete-time systems with disturbances and leverages the efficient reachability calculations of mixed monotone systems to propose an algorithm for computing finite state abstractions, a common prerequisite as noted above for applying formal methods to control systems; specialization of related approaches for the class of monotone systems is studied in the earlier work [20] and more recently in [21], [22]. Mixed monotone systems theory has been extended and applied to traffic flow networks [23], [24], sampled-data systems [25], formal methods [26], [27], sufficient conditions for mixed monotonicity [28], [29], [30], and a toolbox for reachability analysis [31].

The objective of this tutorial is to collect and present recent results, and note connections to older results, in applying mixed monotone systems theory for reachability in a unified way. This tutorial is organized as follows. Section II briefly reviews monotone dynamical systems. Mixed monotonicity in continuous-time is presented in Section III, and the embedding system and its properties are studied in Section IV. The foundational tools for reachability and safety analysis are presented in Section V. Analagous results for discrete-time systems are collected in Section VI, and several case studies are discussed in Section VII. The proofs of some key results are provided here; the proofs of the remaining results are found in the related literature as indicated.

Notation

Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} \mid x \leq 0\}$. Denote the extended reals by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and let $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\overline{\mathbb{R}}_{\leq 0} = \mathbb{R} \cup \{-\infty\}$. Subscript denotes vector index so that a vector $x \in \mathbb{R}^n$ is written as $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ or $x = (x_1, x_2, \dots, x_n)$. For $x, y \in \mathbb{R}^n$, we sometimes write $(x, y) = \begin{bmatrix} x^T & y^T \end{bmatrix}^T \in \mathbb{R}^{2n}$ for the concatenation of x and y. Throughout, the standard inequality symbols $\leq$, $\geq$, $<$ and $>$ are interpreted component-wise. Alternative partial orders will be indicated with subscript. Specifically, we make particular use of the *southeast* order on $\mathbb{R}^{2n}$ denoted $\leq_{\text{SE}}$ and defined as follows: For $(x, \widehat{x}), (y, \widehat{y}) \in \mathbb{R}^{2n}$, $(x, \widehat{x}) \leq_{\text{SE}} (y, \widehat{y})$ if and only if $x \leq y$ and $\widehat{y} \leq \widehat{x}$.

A set $A \subseteq \mathbb{R}^n$ is an *extended hyperrectangle* if there exists $a, b \in \overline{\mathbb{R}}^n$ with $a \leq b$ such that $A = [a, b] := \{x \in \mathbb{R}^n \mid a \leq x \leq b\}$, where the partial order $\leq$ extends to $\overline{\mathbb{R}}^n$ in the natural way. If an extended hyperrectangle is bounded, it is called a *hyperrectangle*. For $a = (x, \widehat{x})$ with $x \leq \widehat{x}$, we write $[\![a]\!] = [x, \widehat{x}]$, *i.e.*, $[\![a]\!]$ for $a \in \mathbb{R}^{2n}$ denotes the hyperrectangle defined by the first and last n components of a. It is a straightforward observation that for any $a = (x, \widehat{x}) \in \mathbb{R}^{2n}$ and $b = (y, \widehat{y}) \in \mathbb{R}^{2n}$ such that $x \leq \widehat{x}$ and $y \leq \widehat{y}$, if $a \leq_{\text{SE}} b$, then $[\![b]\!] \subseteq [\![a]\!]$, equivalently, $[y, \widehat{y}] \subseteq [x, \widehat{x}]$.

II. A Brief Review of Monotone Dynamical Systems

Before defining mixed monotonicity, we recall key results from the theory of monotone dynamical systems. A dynamical system is monotone if it maintains a partial order of states along the evolution of the system state. In particular, we present necessary and sufficient conditions for monotonicity in terms of the sign structure of appropriate Jacobian matrices from the system dynamics. Then, rather than defining mixed monotonicity in terms of the system evolution, it will prove more natural to extend these necessary and sufficient Jacobian-based conditions in order to define mixed monotonicity, and we will show that such an extension implies certain conditions on system flow in a particular *embedding space* that generalize monotonicity.

Monotonicity (and mixed monotonicity discussed subsequently) can be defined with respect to partial orders induced by arbitrary positive cones in a Banach space; however, in this tutorial, we will focus primarily on dynamical systems with state space $\mathcal{X} \subset \mathbb{R}^n$ and the standard partial order on $\mathbb{R}^n$ given by component-wise inequality; the key exception is when we also consider systems with state space a subset of $\mathbb{R}^{2n}$ and the southeast partial order.

We first consider continuous-time dynamical systems and then, in Section VI, we collect analogous results for the discrete-time case. Consider

$$\dot{x} = f(x, w) \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $w \in \mathcal{W} \subseteq \mathbb{R}^m$. We assume throughout that $\mathcal{X}$ is an extended hyperrectangle with nonempty interior and $\mathcal{W}$ is a hyperrectangle, and in particular there exists $\underline{w}, \overline{w} \in \mathbb{R}^m$ such that $\mathcal{W} = [\underline{w}, \overline{w}]$.

Given $\mathbf{w} : [0, \infty) \to \mathcal{W}$, let $\phi(t, x_0, \mathbf{w})$ be the flow map for (1) denoting the state of the system at time t when initialized at state x_0 at time 0 subject to the input $\mathbf{w}$. Throughout, we always assume $\mathbf{w}$ and other time-varying signals are piecewise continuous and that f is locally Lipschitz in x and w so that for any $x_0 \in \mathcal{X}$ and all piecewise continuous signals $\mathbf{w}$, $\phi(t, x_0, \mathbf{w})$ is unique when it exists. There is no need to assume that trajectories remain within the domain $\mathcal{X}$ for all time or that finite escape time is avoided, but $\phi(t, x_0, \mathbf{w})$ is understood to exist only if $\phi(\tau, x_0, \mathbf{w}) \in \mathcal{X}$ for all $\tau \in [0, t]$, *i.e.*, solutions cease to exist upon exiting $\mathcal{X}$. Throughout this tutorial, statements involving the flow map of a dynamical system are understood to be valid only for times at which the flow map exists. In the case that f does not depend on w so that (1) becomes $\dot{x} = f(x)$, we omit $\mathbf{w}$ from the flow map and instead write $\phi(t, x_0)$.

Formally, the system (1) is *monotone* if for any pair $x_0, x_0' \in \mathcal{X}$ such that $x_0 \leq x_0'$ and any pair $\mathbf{w}, \mathbf{w}' : [0, \infty) \to \mathcal{W}$ such that $\mathbf{w}(t) \leq \mathbf{w}'(t)$ for all $t \geq 0$, it holds that $\phi(t, x_0, \mathbf{w}) \leq \phi(t, x_0', \mathbf{w}')$ for all $t \geq 0$. When specialized to the case of the standard partial order, as is the case in this tutorial, monotone systems are also called *cooperative* systems.

Necessary and sufficient conditions for monotonicity given in terms of Jacobian matrices from the system dynamics are given as follows.

Proposition 1 ([10, Proposition III.2])**.** *The system* (1) *is monotone if and only if the following hold:*

- *For all $i, j \in \{1, \ldots, n\}$ with $i \neq j$,*
$$\frac{\partial f_i}{\partial x_j}(x, w) \geq 0 \tag{2}$$
for all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$ whenever the derivative exists.
- *For all $i \in \{1, \ldots, n\}$ and all $k \in \{1, \ldots, m\}$,*
$$\frac{\partial f_i}{\partial w_k}(x, w) \geq 0 \tag{3}$$
for all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$ whenever the derivative exists.

Since f is assumed to be Lipschitz, the above derivatives exist almost everywhere, *i.e.*, except on a set of measure zero.

It is appropriate to think of the conditions in Proposition 1 as an *infinitesimal* characterization of monotonicity. In the next section, we extend the infinitesimal characterization of Proposition 1 to define mixed monotone dynamical systems.

III. Mixed Monotone Dynamical Systems in Continuous-Time

Mixed monotonicity generalizes monotonicity by decomposing a system into its increasing and decreasing components. We define mixed monotonicity be extending the Jacobian condition above to a *decomposition function* that must be related to the dynamics in a particular way.

Definition 1. Given a locally Lipschitz continuous function $d : \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$, the system (1) is *mixed monotone with respect to* d if the following hold:

- For all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$,
$$d(x, w, x, w) = f(x, w). \tag{4}$$
- For all $i, j \in \{1, \ldots, n\}$ with $i \neq j$,
$$\frac{\partial d_i}{\partial x_j}(x, w, \widehat{x}, \widehat{w}) \geq 0 \tag{5}$$
for all $x, \widehat{x} \in \mathcal{X}$ and for all $w, \widehat{w} \in \mathcal{W}$ whenever the derivative exists.
- For all $i, j \in \{1, \ldots, n\}$,
$$\frac{\partial d_i}{\partial \widehat{x}_j}(x, w, \widehat{x}, \widehat{w}) \leq 0 \tag{6}$$
for all $x, \widehat{x} \in \mathcal{X}$ and for all $w, \widehat{w} \in \mathcal{W}$ whenever the derivative exists.
- For all $i \in \{1, \ldots, n\}$ and all $k \in \{1, \ldots, m\}$,
$$\frac{\partial d_i}{\partial w_k}(x, w, \widehat{x}, \widehat{w}) \geq 0 \text{ and } \frac{\partial d_i}{\partial \widehat{w}_k}(x, w, \widehat{x}, \widehat{w}) \leq 0 \tag{7}$$
for all $x, \widehat{x} \in \mathcal{X}$ and for all $w, \widehat{w} \in \mathcal{W}$ whenever the derivative exists.

■

If (1) is mixed monotone with respect to d, d is said to be a *decomposition function* for (1), and when d is clear from context, we simply say (1) is *mixed monotone*. When there is no disturbance so that $\dot{x} = f(x)$, we instead write the decomposition function as $d(x, \widehat{x})$. Note the asymmetry between condition (6), which must hold for all i, j, and condition (5), which need only hold for $i \neq j$.

The definition of mixed monotonicity is in terms of the derivatives of the decomposition function d. By integrating, mixed monotonicity can be equivalently characterized as in Fact 1 below, the proof of which is analogous to the proof of Proposition 1; we call these the *Kamke-type* conditions for mixed monotonicity since they are the analog of the Kamke conditions for monotonicity as discussed in, *e.g.*, [11, Ch. 3].

Fact 1 (Kamke-type conditions)**.** *The system* (1) *is mixed monotone with respect to* d *if and only if the following hold:*

- *For all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$,*
$$d(x, w, x, w) = f(x, w). \tag{8}$$
- *For all $i \in \{1, \ldots, n\}$,*
$$d_i(x, w, \widehat{x}, \widehat{w}) \leq d_i(y, v, \widehat{x}, \widehat{w}) \tag{9}$$
for all $x, y, \widehat{x} \in \mathcal{X}$ such that $x \leq y$ and $x_i = y_i$ and for all $w, v, \widehat{w} \in \mathcal{W}$ such that $w \leq v$.
- *For all $i \in \{1, \ldots, n\}$,*
$$d_i(x, w, \widehat{y}, \widehat{v}) \leq d_i(x, w, \widehat{x}, \widehat{w}) \tag{10}$$
for all $x, \widehat{x}, \widehat{y} \in \mathcal{X}$ such that $\widehat{x} \leq \widehat{y}$ and for all $w, \widehat{w}, \widehat{v} \in \mathcal{W}$ such that $\widehat{w} \leq \widehat{v}$.

Mixed monotonicity generalizes monotonicity.

Example 1. The system (1) is monotone if and only if it is mixed monotone with respect to the particular decomposition function defined by $d(x, w, \widehat{x}, \widehat{w}) = f(x, w)$ for all $x, \widehat{x} \in \mathcal{X}$ and all $w, \widehat{w} \in \mathcal{W}$. In this case, the conditions given in Definition 1 reduce to the infinitesimal characterization of monotonicity given in Proposition 1. ■

We make a few observations regarding Definition 1. Monotone dynamical systems are defined as those systems whose flow satisfies a certain monotonicity property; in contrast, Definition 1 makes no mention of the flow of any dynamical system. It turns out that it is easier to define mixed monotonicity by appealing to its infinitesimal characterization given in terms of certain Jacobian matrices, and then we will see in Section IV that the appropriate connection is to the flow of a particular $2n$ dimensional embedding system. Moreover, the only restriction on d imposed by f is given in (4). This suggests that a system (1) may be mixed monotone with respect to many decomposition functions, and, thus, we include "with respect to d" in the definition above to emphasize this critical dependence. In fact, there always exists a decomposition function, as established next.

Theorem 1 ([30, Theorem 1])**.** *Any system of the form* (1) *is mixed monotone with respect to a decomposition function* d *satisfying*

$$d_i(x, w, \widehat{x}, \widehat{w}) = \begin{cases} \min\limits_{\substack{y \in [x, \widehat{x}] \\ y_i = x_i \\ z \in [w, \widehat{w}]}} f_i(y, z) & \text{if } x \leq \widehat{x} \text{ and } w \leq \widehat{w}, \\ \max\limits_{\substack{y \in [\widehat{x}, x] \\ y_i = x_i \\ z \in [\widehat{w}, w]}} f_i(y, z) & \text{if } \widehat{x} \leq x \text{ and } \widehat{w} \leq w. \end{cases} \tag{11}$$

Note that, in the above theorem, d is only specified for $x, \widehat{x}, w, \widehat{w}$ such that $x \leq \widehat{x}$ and $w \leq \widehat{w}$, or $\widehat{x} \leq x$ and $\widehat{w} \leq w$. As we will see in the following section, the values that d takes outside of this region are not relevant for the purposes of analysis of (1), and it is in fact possible to restrict the domain of a decomposition function d to this region without loss of generality, as is done in [30]. Moreover, in addition to characterizing some decomposition function, the decomposition function given in (11) is the tightest decomposition function in the sense that it provides the smallest reachable set computations available by Proposition 3 discussed subsequently; see [30] for further details.

In some cases, there exists a closed-form decomposition function satisfying the conditions of (11).

Example 2. Consider the system $\dot{x} = f(x)$ with

$$f(x) = \begin{bmatrix} x_2^2 + 2 \\ x_1 \end{bmatrix} \tag{12}$$

and $\mathcal{X} = \mathbb{R}^2$. This system is mixed monotone with decomposition function

$$d_1(x, \widehat{x}) = \begin{cases} x_2^2 + 2 & \text{if } x_2 \geq 0 \text{ and } x_2 \geq -\widehat{x}_2 \\ \widehat{x}_2^2 + 2 & \text{if } \widehat{x}_2 \leq 0 \text{ and } x_2 < -\widehat{x}_2 \\ 2 & \text{if } x_2 < 0 \text{ and } \widehat{x}_2 > 0, \end{cases} \tag{13}$$
$$d_2(x, \widehat{x}) = x_1$$

that satisfies (11). ■

Despite the apparent generality of Theorem 1, solving the optimization problems specified in (11) is generally not how decomposition functions are obtained in practice. Indeed, from the perspective of chronological developments in the theory of mixed monotone systems, Theorem 1 and its discrete-time counterpart Theorem 3 are recent results; more commonly, decomposition functions are constructed from domain knowledge of the problem at hand or via certain special cases such as those described below.

There are several reasons why Theorem 1 is of limited practical use. First, solutions to the optimization in (11) may not be available in closed-form. Second, even when a solution is available in closed-form, it is often defined piecewise as in Example 2, and the number of pieces generally scales exponentially with the dimension of the state and disturbance space, and thus evaluating the decomposition function at a particular point may require a large number of function evaluations. As we will see below, a main appeal of mixed monotone systems theory is its computational tractability: computing reachable sets for an n-dimensional system requires computing a single trajectory of a $2n$-dimensional system so that reachable set computations scale apparently linearly in state dimension. But this higher dimensional system is defined from the decomposition function, and thus computing, *e.g.*, simulated trajectories would require an exponential number of condition evaluations at each time-step if the decomposition function contains an exponential number of pieces.

Aside from obtaining decomposition functions using domain knowledge, an important special case that leads to a particularly simple decomposition function is when each off-diagonal entry of the Jacobian matrix $\partial f / \partial x$ and each entry of the Jacobian matrix $\partial f / \partial w$ is either upper or lower bounded, as formalized in the following special case.

Special Case 1. Consider (1). If there exists $\underline{J}_x \in \overline{\mathbb{R}}^{n \times n}_{\leq 0}$, $\overline{J}_x \in \overline{\mathbb{R}}^{n \times n}_{\geq 0}$, $\underline{J}_w \in \overline{\mathbb{R}}^{n \times m}_{\leq 0}$, and $\overline{J}_w \in \overline{\mathbb{R}}^{n \times m}_{\geq 0}$ such that

- for all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$,
$$\frac{\partial f}{\partial x}(x, w) \in [\underline{J}_x, \overline{J}_x] \text{ and } \frac{\partial f}{\partial w}(x, w) \in [\underline{J}_w, \overline{J}_w] \tag{14}$$
whenever the derivative exists, and
- for all $i \neq j$, $(\underline{J}_x)_{i,j} > -\infty$ or $(\overline{J}_x)_{i,j} < \infty$, and
- for all i, k, $(\underline{J}_w)_{i,k} > -\infty$ or $(\overline{J}_w)_{i,k} < \infty$

then (1) is mixed monotone and a decomposition function is constructed in the following way:

1) For all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and all $k \in \{1, \ldots, m\}$, choose $\delta_{i,j}, \epsilon_{i,k} \in \{0, 1\}$ such that
$$\begin{array}{lll} \delta_{i,j} = 0 & \text{implies} & (\underline{J}_x)_{i,j} \neq -\infty, \\ \delta_{i,j} = 1 & \text{implies} & (\overline{J}_x)_{i,j} \neq \infty, \\ \epsilon_{i,k} = 0 & \text{implies} & (\underline{J}_w)_{i,k} \neq -\infty, \\ \epsilon_{i,k} = 1 & \text{implies} & (\overline{J}_w)_{i,k} \neq \infty. \end{array} \tag{15}$$
Note that such a choice exists by hypothesis.
2) For all $i \in \{1, \ldots, n\}$, define $\xi^i, \alpha^i \in \mathbb{R}^n$ and $\pi^i, \beta^i \in \mathbb{R}^m$ element-wise according to
$$(\xi^i_i, \alpha^i_i) = (x_i, 0), \tag{16}$$
$$(\xi^i_j, \alpha^i_j) = \begin{cases} (x_j, -(\underline{J}_x)_{i,j}) & \text{if } i \neq j \text{ and } \delta_{i,j} = 0, \\ (\widehat{x}_j, (\overline{J}_x)_{i,j}) & \text{if } i \neq j \text{ and } \delta_{i,j} = 1, \end{cases} \tag{17}$$
$$(\pi^i_k, \beta^i_k) = \begin{cases} (w_k, -(\underline{J}_w)_{i,k}) & \text{if } \epsilon_{i,k} = 0, \\ (\widehat{w}_k, (\overline{J}_w)_{i,k}) & \text{if } \epsilon_{i,k} = 1. \end{cases} \tag{18}$$

3) Lastly, define the i^{th} element of the decomposition function d according to

$$d_i(x, w, \widehat{x}, \widehat{w}) = f_i(\xi^i, \pi^i) + (\alpha^i)^T(x - \widehat{x}) + (\beta^i)^T(w - \widehat{w}), \quad (19)$$

which is always well-defined on $\mathcal{X}\times\mathcal{W}\times\mathcal{X}\times\mathcal{W}$ since $\mathcal{X}$ and $\mathcal{W}$ are assumed to be extended hyperrectangles. ■

Note that a monotone system satisfying Proposition 1 satisfies the hypothesis of Special Case 1 so that the choice $\delta_{i,j} = 0$ for all $i \neq j$ and $\epsilon_{i,k} = 0$ for all i and k and taking $\underline{J}_x = 0$ and $\underline{J}_w = 0$ leads to the canonical decomposition function for monotone systems noted in Example 1. In addition, any linear system $\dot{x} = Ax + Bw$ also satisfies the hypotheses of Special Case 1, as shown next.

Example 3. Any linear system $\dot{x} = Ax + Bw$ satisfies the hypotheses of Special Case 1 and is therefore mixed monotone. In particular, write A and B as $A = A^1 - A^2$ and $B = B^1 - B^2$ where $A^1_{i,i} = A_{i,i}$ for all i, $A^1_{i,j} = \max\{0, A_{i,j}\}$ for $i \neq j$, $A^2 = A^1 - A$, $B^1 = \max\{0, B\}$ and $B^2 = \max\{0, -B\}$ where the max is understood to be elementwise. Then $d(x, w, \widehat{x}, \widehat{w}) = A^1x + B^1w - (A^2\widehat{x} + B^2\widehat{w})$ is a decomposition function for the system. This is the decomposition function given by the procedure of Special Case 1 with the choice $\delta_{i,j} = 0$ if and only if $A_{i,j} \geq 0$ for each $i \neq j$ and $\epsilon_{i,k} = 0$ if and only if $B_{i,k} \geq 0$ and taking $\underline{J}_x = -A^2$, $\overline{J}_x = A^1$, $\underline{J}_w = -B^2$, and $\overline{J}_w = B^1$. ■

The next example is inspired by [17, Lemma 1].

Example 4. In Special Case 1, if no off-diagonal entries of $\underline{J}_x$ and no entries of $\underline{J}_w$ take the value $-\infty$, then taking $\delta_{i,j} = 0$ for all $i \neq j$ and $\epsilon_{i,k} = 0$ for all i, k leads to the decomposition function $d(x, w, \widehat{x}, \widehat{w}) = f(x, w) - \underline{J}_x(x - \widehat{x}) - \underline{J}_w(w - \widehat{w})$ where, without loss of generality, we set $(\underline{J}_x)_{i,i} = 0$ for all i for this example (note that no bound on $\partial f_i/\partial x_i$ for any i is needed or used in the construction of the decomposition function in Special Case 1). This decomposition function separates the dynamics as the difference of two monotone systems with $f(x, w) = f^1(x, w) - f^2(x, w)$, $f^1(x, w) = f(x, w) - \underline{J}_x x - \underline{J}_w w$ and $f^2(x, w) = -\underline{J}_x x - \underline{J}_w w$, and decomposition function $d(x, w, \widehat{x}, \widehat{w}) = f^1(x, w) - f^2(\widehat{x}, \widehat{w})$. ■

Even though Special Case 1 encompasses a large class of systems, some systems do not satisfy the conditions stipulated by the special case but are nonetheless mixed monotone with respect to an alternative decomposition function. This is a main observation of [32].

Example 5 (Example 2 continued)**.** Consider again the system $\dot{x} = f(x)$ with $f(x)$ as in (12) and $\mathcal{X} = \mathbb{R}^2$. Note that $\frac{\partial f_1}{\partial x_2}(x) = 2x_2$ is neither lower or upper bounded on $\mathcal{X}$ and thus the system does not satisfy the hypotheses of Special Case 1, yet a decomposition function is provided in Example 2.

Now consider a restricted domain defined on some compact subset $\mathcal{X}' \subset \mathcal{X}$. Then the decomposition function construction defined in Special Case 1 is applicable on the restricted domain $\mathcal{X}'$. For instance, take $\mathcal{X}' = [-5, 5] \times [-5, 5]$. On $\mathcal{X}'$, $\frac{\partial f}{\partial x} \in [\underline{J}_x, \overline{J}_x]$ with

$$\underline{J}_x = \begin{bmatrix} 0 & -10 \\ 1 & 0 \end{bmatrix} \qquad \overline{J}_x = \begin{bmatrix} 0 & 10 \\ 1 & 0 \end{bmatrix} \quad (20)$$

so that Special Case 1 with $\delta_{1,2} = \delta_{2,1} = 1$ gives the alternative decomposition function

$$d'(x, \widehat{x}) = \begin{bmatrix} \widehat{x}_2^2 + 2 + 10(x_2 - \widehat{x}_2) \\ x_1 \end{bmatrix}. \quad (21)$$

■

We conclude this section by discussing the relationship and differences between Definition 1, Special Case 1, and other definitions and conditions appearing in the literature. A version of Definition 1 specialized to the case with no disturbances appears in [24, Definition 1], and a restrictive version of Special Case 1 requiring sign-stability of the Jacobian matrices is noted in [33]. The essential observation that the special case extends to the case when the entries of the Jacobian matrices are bounded is made in [26], [28], [31].

However, instead of viewing Special Case 1 as a special case of Definition 1, [26] and [31] instead define a mixed monotone system to be one that satisfies the conditions of Special Case 1. In addition, the conditions given in [26], [28], [31] differ from that in Special Case 1 in several minor ways. First, it is often preferable (for, e.g., reachability analysis) to choose $\delta_{i,j} = 0$ whenever $(\underline{J}_x)_{i,j} + (\overline{J}_x)_{i,j} \geq 0$ and to choose $\delta_{i,j} = 1$ otherwise, and similarly to choose $\epsilon_{i,k} = 0$ if and only if $(\underline{J}_w)_{i,k} + (\overline{J}_w)_{i,k} \geq 0$. This choice is imposed in the conditions given in [26], [28], [31] but is not necessary for arriving at a valid decomposition function and Special Case 1 does not impose this requirement. Moreover, [31, Assumption 5] imposes that each off-diagonal entry of $\underline{J}_x$ and $\overline{J}_x$ and each entry of $\underline{J}_w$ and $\overline{J}_w$ be finite, but as noted in [28], these entries need only be upper or lower bounded. Lastly, [26], [28] further imposes that the diagonal entries of $\partial f/\partial x$ are also bounded, but this is not needed and, in particular, precludes the Special Case from encompassing monotonicity as characterized in Proposition 1. Also of note is [34], which studies the response to monotone inputs of systems with sign-stable Jacobian matrices.

IV. The Embedding System in Continuous-Time

Separating the dynamics of (1) into increasing and decreasing components with a decomposition function as in Definition 1 allows the dynamics of (1) to be embedded into a higher dimensional space with advantageous monotonicity properties. In this section, we construct two *embedding systems* constructed from a decomposition function d for a mixed monotone system. The first is an embedding system with disturbance for which the theory of monotone control systems as developed in [10] is applicable. By fixing the disturbance at extreme values, we obtain a deterministic embedding system that deserves study in its own right.

Consider system (1) that is mixed monotone with respect to decomposition function d. We first construct a $2n$ dimensional dynamical system subject to a $2m$ dimensional disturbance vector as follows. Consider

$$\begin{bmatrix} \dot{x} \\ \dot{\widehat{x}} \end{bmatrix} = \varepsilon(x, w, \widehat{x}, \widehat{w}) := \begin{bmatrix} d(x, w, \widehat{x}, \widehat{w}) \\ d(\widehat{x}, \widehat{w}, x, w) \end{bmatrix} \tag{22}$$

with state $(x, \widehat{x}) \in \mathcal{X} \times \mathcal{X} \subset \mathbb{R}^{2n}$ and disturbance input $(w, \widehat{w}) \in \mathcal{W} \times \mathcal{W} \subset \mathbb{R}^{2m}$. Given piecewise continuous inputs $\mathbf{w}, \widehat{\mathbf{w}} : [0, \infty) \to \mathcal{W}$, let $\Phi^{\varepsilon}(t, (x_0, \widehat{x}_0), (\mathbf{w}, \widehat{\mathbf{w}}))$ denote the state of (22) at time t when initialized at $(x_0, \widehat{x}_0) \in \mathcal{X} \times \mathcal{X}$ under the inputs $\mathbf{w}$ and $\widehat{\mathbf{w}}$. As in the case for solutions to (1), for $a \in \mathcal{X} \times \mathcal{X}$ and $\mathbf{b} : [0, \infty) \to \mathcal{W} \times \mathcal{W}$, $\Phi^{\varepsilon}(t, a, \mathbf{b})$ is understood to only exist if $\Phi^{\varepsilon}(\tau, a, \mathbf{b}) \in \mathcal{X} \times \mathcal{X}$ for all $\tau \in [0, t]$, and all statements involving the flow of (22) are understood to be valid only when the flow map exists.

Recall the southeast partial order on $\mathbb{R}^{2n}$ with the defining property that for $(x, \widehat{x}), (y, \widehat{y}) \in \mathbb{R}^{2n}$, $(x, \widehat{x}) \leq_{\text{SE}} (y, \widehat{y})$ if and only if $x \leq y$ and $\widehat{y} \leq \widehat{x}$. It is called the southeast order because, for $n = 1$, when considered graphically, $(x, \widehat{x}) \leq_{\text{SE}} (y, \widehat{y})$ whenever $(y, \widehat{y})$ is to the lower-right, *i.e.*, southeast, of $(x, \widehat{x})$. We also use the same notation $\leq_{\text{SE}}$ to denote the southeast partial order on $\mathbb{R}^{2m}$.

The critical observation is that (22) is a monotone system with respect to the southeast orders on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$.

Proposition 2. *Suppose* (1) *is mixed monotone with respect to d. For any $a, a' \in \mathcal{X} \times \mathcal{X}$ and $\mathbf{b}, \mathbf{b}' : [0, \infty) \to \mathcal{W} \times \mathcal{W}$, if $a \leq_{\text{SE}} a'$ and $\mathbf{b}(t) \leq_{\text{SE}} \mathbf{b}'(t)$ for all $t \geq 0$, then*

$$\Phi^{\varepsilon}(t, a, \mathbf{b}) \leq_{\text{SE}} \Phi^{\varepsilon}(t, a', \mathbf{b}') \tag{23}$$

for all $t \geq 0$.

Proof. From (22), define new state variables $z = -\widehat{x}$ and $v = -\widehat{w}$ after a coordinate transformation. Then the southeast order in the original $(x, \widehat{x})$ and $(w, \widehat{w})$ coordinates becomes the standard coordinate-wise order in the new (x, z) and (w, v) coordinates. Then, it is straightforward to verify that the conditions on d given in (5)–(7) implies the conditions of Proposition 1 in the new (x, z) state coordinates and (w, v) disturbance input coordinates. □

Recall our original aim of reachability and safety analysis for (1), which is concerned with identifying forward invariant sets, as formalized in the following definition.

Definition 2. A set $A \subseteq \mathcal{X}$ is *robustly forward invariant* for (1) if $\phi(T, x_0, \mathbf{w}) \in A$ for all $x_0 \in A$, all $T \geq 0$ and all piecewise continuous inputs $\mathbf{w} : [0, T] \to \mathcal{W}$ whenever $\phi(T, x_0, \mathbf{w})$ exists. When f does not depend on w, A is said to be *forward invariant*. ■

Remark 1. Recall that solutions to (1) cease to exist upon exiting $\mathcal{X}$. Thus, the domain $\mathcal{X}$ itself is always (vacuously) robustly forward invariant. If, instead, f in (1) is interpreted as being defined for all $x \in \mathbb{R}^n$ and, likewise, ε in (22) as defined for all $(x, \widehat{x}) \in \mathbb{R}^{2n}$, there is no guarantee that $\mathcal{X} \times \mathcal{X}$ is robustly forward invariant for (22) even if $\mathcal{X}$ is robustly forward invariant for (1). Thus it is important to note the validity of (23) only when $\Phi^{\varepsilon}(t, a, \mathbf{b})$ and $\Phi^{\varepsilon}(t, a', \mathbf{b}')$ exist, *i.e.*, when the system remains in $\mathcal{X} \times \mathcal{X}$ on $[0, t]$. ■

System (22) satisfies the following symmetry property: For any $x_0, \widehat{x}_0 \in \mathcal{X}$ and $\mathbf{w}, \widehat{\mathbf{w}} : [0, \infty) \to \mathcal{W}$, letting $(x(t), \widehat{x}(t)) = \Phi^{e}(t, (x_0, \widehat{x}_0), (\mathbf{w}, \widehat{\mathbf{w}}))$, it holds that $(\widehat{x}(t), x(t)) = \Phi^{e}(t, (\widehat{x}_0, x_0), (\widehat{\mathbf{w}}, \mathbf{w}))$.

Recall the disturbance set $\mathcal{W} = [\underline{w}, \overline{w}]$ for (1). By fixing the disturbances $\mathbf{w} \equiv \underline{w}$ and $\widehat{\mathbf{w}} \equiv \overline{w}$, we obtain a second embedding system without disturbance. In particular, consider

$$\begin{bmatrix} \dot{x} \\ \dot{\widehat{x}} \end{bmatrix} = e(x, \widehat{x}) := \begin{bmatrix} d(x, \underline{w}, \widehat{x}, \overline{w}) \\ d(\widehat{x}, \overline{w}, x, \underline{w}) \end{bmatrix} \tag{24}$$

and denote its solutions by $\Phi^{e}(t, (x_0, \widehat{x}_0))$ for initial condition $(x_0, \widehat{x}_0) \in \mathcal{X} \times \mathcal{X}$, which is assumed to exist only when $\Phi^{e}(\tau, (x_0, \widehat{x}_0)) \in \mathcal{X} \times \mathcal{X}$ for $\tau \in [0, t]$.

Corollary 1. *Suppose* (1) *is mixed monotone with respect to d. For any $a, a' \in \mathcal{X} \times \mathcal{X}$, if $a \leq_{\text{SE}} a'$, then*

$$\Phi^{e}(t, a) \leq_{\text{SE}} \Phi^{e}(t, a') \tag{25}$$

for all $t \geq 0$.

The utility of mixed monotone systems theory stems from Proposition 2 and Corollary 1. In particular, embedding the dynamics of (1) into a monotone embedding system (22) or (24) enables using the powerful theory of monotone systems to study its behavior.

We begin with some fundamental invariance properties of (22) and (24). Define

$$\mathcal{D} = \{(x, \widehat{x}) \in \mathcal{X} \times \mathcal{X} \mid x = \widehat{x}\}, \tag{26}$$

$$\mathcal{T} = \{(x, \widehat{x}) \in \mathcal{X} \times \mathcal{X} \mid x \leq \widehat{x}\}. \tag{27}$$

The set $\mathcal{D}$ is called the *diagonal* and $\mathcal{T}$ the *upper triangle* for (22) and (24). The next lemma establishes that both $\mathcal{D}$ and $\mathcal{T}$ exhibit invariance-like properties for (22).

Lemma 1. *Suppose* (1) *is mixed monotone with respect to d. For any $\mathbf{w}_1, \mathbf{w}_2 : [0, \infty) \to \mathcal{W}$, if $\mathbf{w}_1(t) \leq \mathbf{w}_2(t)$ for all $t \geq 0$ and $(x_0, \widehat{x}_0) \in \mathcal{T}$, then $\Phi^{\varepsilon}(t, (x_0, \widehat{x}_0), (\mathbf{w}_1, \mathbf{w}_2)) \in \mathcal{T}$ for all $t \geq 0$. Moreover, for any $\mathbf{w} : [0, \infty) \to \mathcal{W}$, if $(x_0, \widehat{x}_0) \in \mathcal{D}$, then $\Phi^{\varepsilon}(t, (x_0, \widehat{x}_0), (\mathbf{w}, \mathbf{w})) \in \mathcal{D}$ for all $t \geq 0$.*

Proof. Let $(x_0, \widehat{x}_0) \in \mathcal{T}$ and note that $(x_0, \widehat{x}_0) \leq_{\text{SE}} (\widehat{x}_0, x_0)$. Moreover, setting $(x(t), \widehat{x}(t)) := \Phi^{\varepsilon}(t, (x_0, \widehat{x}_0), (\mathbf{w}_1, \mathbf{w}_2))$, we have also $(\widehat{x}(t), x(t)) = \Phi^{\varepsilon}(t, (\widehat{x}_0, x_0), (\mathbf{w}_2, \mathbf{w}_1))$ due to the symmetry of (22), where $\mathbf{w}_1$ and $\mathbf{w}_2$ are any two functions satisfying the hypotheses of the lemma. Since $(\mathbf{w}_1(t), \mathbf{w}_2(t)) \leq_{\text{SE}} (\mathbf{w}_2(t), \mathbf{w}_1(t))$ for all $t \geq 0$, by Proposition 2, $\Phi^{\varepsilon}(t, (x_0, \widehat{x}_0), (\mathbf{w}_1, \mathbf{w}_2)) \leq_{\text{SE}} \Phi^{\varepsilon}(t, (\widehat{x}_0, x_0), (\mathbf{w}_2, \mathbf{w}_1))$ so that $x(t) \leq \widehat{x}(t)$, *i.e.*, $(x(t), \widehat{x}(t)) \in \mathcal{T}$, for all $t \geq 0$. The second part follows by noting that when, in addition, $(x_0, \widehat{x}_0) \in \mathcal{D}$ and $w = \widehat{w}$ always, (22) reduces to two copies of the original system (1), *i.e.*, the first and last n components of (22) both reduce to the dynamics of the original system. □

For (24), we have the following corollary.

Corollary 2. *Suppose* (1) *is mixed monotone with respect to* d*. If* $(x_0, \widehat{x}_0) \in \mathcal{T}$*, then* $\Phi^e(t, (x_0, \widehat{x}_0)) \in \mathcal{T}$ *for all* $t \geq 0$*.*

Proof. This follows from the first part of Lemma 1 by taking $\mathbf{w}_1 \equiv \underline{w}$ and $\mathbf{w}_2 \equiv \overline{w}$. □

As we will see in the next section, the dynamic behavior of (22) and (24) within $\mathcal{T}$ translates to reachable set computations for the original system (1), and equilibria and their stability properties for (24) enable identifying robustly invariant sets for (1).

Lastly, we note that when no disturbance is present, the embedding systems (22) and (24) reduce to the same system and $\mathcal{D}$ is a forward invariant set for this system. This is because, in this case, the embedding dynamics confined to $\mathcal{D}$ consist of two copies of the original dynamics.

Corollary 3. *Suppose* (1) *is mixed monotone with respect to* d*. When no disturbance is present so that* (1) *reduces to* $\dot{x} = f(x)$*, the diagonal* $\mathcal{D}$ *is forward invariant for* (24)*.*

V. Reachability and Safety for Continuous-Time Mixed Monotone Systems

For $\mathcal{X}' \subseteq \mathcal{X}$, denote the reachable set at time T of (1) when initialized within $\mathcal{X}'$ by

$$R^f(T, \mathcal{X}') = \{\phi(T, x_0, \mathbf{w}) \mid x_0 \in \mathcal{X}' \text{ and } \mathbf{w}(t) \in \mathcal{W} \ \forall t \geq 0\}. \quad (28)$$

The next result provides the fundamental connection between reachable sets and the dynamics of embedding systems.

Proposition 3. *Suppose* (1) *is mixed monotone with respect to* d*. Given an initial hyperrectangle of states* $\mathcal{X}_0 = [\underline{x}, \overline{x}] \subseteq \mathcal{X}$ *and disturbance bounds* $\mathbf{w}_1, \mathbf{w}_2 : [0, \infty) \to \mathcal{W}$ *satisfying* $\mathbf{w}_1(t) \leq \mathbf{w}_2(t)$ *for all* $t \geq 0$*,*

$$\phi(T, x_0, \mathbf{w}) \in [\![\Phi^\varepsilon(T, (\underline{x}, \overline{x}), (\mathbf{w}_1, \mathbf{w}_2))]\!] \quad (29)$$

for all $T \geq 0$*, all* $x_0 \in \mathcal{X}_0$*, and all* $\mathbf{w}$ *satisfying* $\mathbf{w}_1(t) \leq \mathbf{w}(t) \leq \mathbf{w}_2(t)$ *for all* $t \geq 0$*. In particular,*

$$R^f(T, [\underline{x}, \overline{x}]) \subseteq [\![\Phi^e(T, (\underline{x}, \overline{x}))]\!] \quad (30)$$

for all $T \geq 0$*.*

Proof. For the first statement, since $(\underline{x}, \overline{x}) \leq_{\mathrm{SE}} (x_0, x_0)$ and $(\underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t)) \leq_{\mathrm{SE}} (\mathbf{w}(t), \mathbf{w}(t))$ for all $t \geq 0$, it follows from Proposition 2 that $\Phi^\varepsilon(T, (\underline{x}, \overline{x}), (\underline{\mathbf{w}}, \overline{\mathbf{w}})) \leq_{\mathrm{SE}} \Phi^\varepsilon(T, (x_0, x_0), (\mathbf{w}, \mathbf{w})) = (\phi(T, x_0, \mathbf{w}), \phi(T, x_0, \mathbf{w}))$, which is equivalent to (29). The second statement follows from the first by taking $\underline{\mathbf{w}}(t) \equiv \underline{w}$ and $\overline{\mathbf{w}}(t) \equiv \overline{w}$ and from the definition of (24). □

Proposition 3, and in particular (30), means that reachable sets for the n dimensional mixed monotone system (1) subject to unknown disturbances can be efficiently over-approximated by computing the solution of a $2n$ dimensional deterministic dynamical system. The proof of the first statement of Proposition 3 is essentially the same as the proof of [31, Proposition 6], and the second statement, which notes a more direct connection to the embedding system (24) without disturbance, is in [32, Proposition 1].

The fundamental observation of Proposition 3 can be further refined to certify safety properties of (1) and, more specifically, to compute sets that are robustly forward invariant and/or *attractive* for (1) as defined next.

Definition 3. A set $A \subset \mathcal{X}$ is *attractive for* (1) *from* $\mathcal{X}' \subset \mathcal{X}$ if for each solution $\phi(\cdot, x_0, \mathbf{w})$ to (1) with $x_0 \in \mathcal{X}'$ and piecewise continuous $\mathbf{w}$ and each relatively open neighborhood $\mathcal{X}_\epsilon \subset \mathcal{X}$ of A, there exists a $T > 0$ such that $\phi(t, x_0, \mathbf{w}) \in \mathcal{X}_\epsilon$ for all $t \geq T$. If A is attractive from some relatively open neighborhood of A, we say A is *locally attractive* or just *attractive*, and when A is attractive from $\mathcal{X}$, we say A is *globally attractive*. ■

The following theorem uses properties of the deterministic embedding system (24) to identify robustly forward invariant and attractive sets for the mixed monotone system (1).

Theorem 2 ([32, Theorem 1])**.** *Suppose* (1) *is mixed monotone with respect to* d*. For any* $a = (x, \widehat{x}) \in \mathcal{T}$ *such that* $0 \leq_{\mathrm{SE}} e(x, \widehat{x})$*, the following hold:*

1) *For all* $T \geq 0$*, the set* $[\![\Phi^e(T, a)]\!] \subseteq \mathcal{X}$ *is robustly forward invariant for* (1)*.*
2) *The limit* $\lim_{t\to\infty} \Phi^e(t, a) =: (x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}})$ *exists and* $e(x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}}) = 0$*,* i.e., $(x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}})$ *is an equilibrium for the deterministic embedding system* (24)*.*
3) *The set* $[x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}}]$ *is robustly forward invariant and attractive from* $[\![a]\!] \subseteq \mathcal{X}$*.*

Theorem (2) can be particularly leveraged if some asymptotically stable equilibrium point of the embedding system (24) is known.

Corollary 4 ([32, Corollary 1])**.** *Suppose* (1) *is mixed monotone with respect to* d*. If* $(x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}}) \in \mathcal{T}$ *is an asymptotically stable equilibrium for* (24) *with a basin of attraction* $\mathcal{C} \subseteq \mathcal{X} \times \mathcal{X}$*, then* $[x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}}]$ *is robustly forward invariant for* (1) *and attractive from any set* $[\![a]\!]$ *such that* $a \in \mathcal{C} \cap \mathcal{T}$*. In particular, if* $\mathcal{T} \subseteq \mathcal{C}$*, then* $[x_{\mathrm{eq}}, \widehat{x}_{\mathrm{eq}}]$ *is globally attractive and robustly forward invariant for* (1)*.*

Example 6 (Examples 2 and 5 continued)**.** Consider again the system $\dot{x} = f(x)$ with $f(x)$ as in (12) and $\mathcal{X} = \mathbb{R}^2$. Figures 1a and 1b demonstrate how Proposition 3 is used to approximate reachable sets for this system. Figure 1c compares reachable sets computed from the tightest decomposition function given in Example 2 and a decomposition function obtained from Special Case 1 as given in Example 5. ■

Example 7 ([32, Example 2])**.** Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = F(x, w) = \begin{bmatrix} -x_1 - x_1^3 - x_2 - w \\ -x_2 - x_2^3 + x_1 + w^3 \end{bmatrix} \quad (31)$$

with $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{W} = [-2, 2]$. This system is mixed

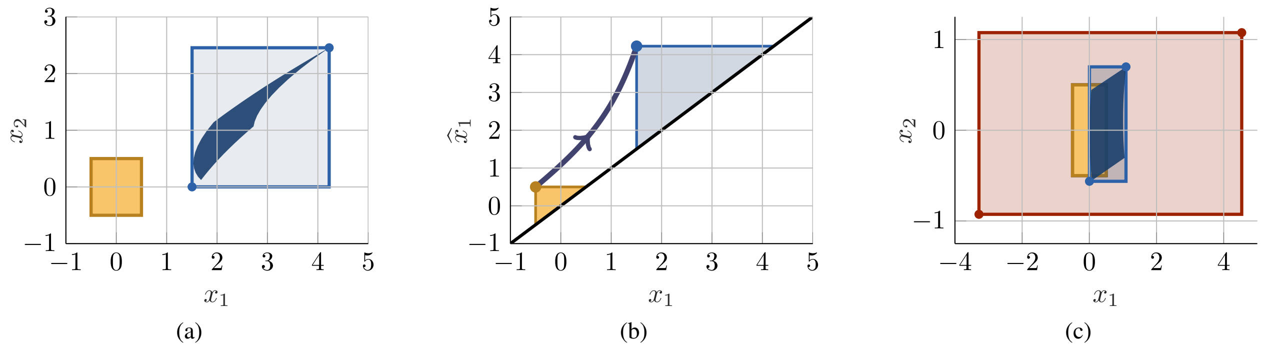

Fig. 1: Approximating forward reachable sets of (12) in Examples 2, 5, and 6 from the set of initial conditions $\mathcal{X}_0 = [-0.5, 0.5] \times [-0.5, 0.5]$. (a) $\mathcal{X}_0$ is shown in orange (medium shading). $R^F(1, \mathcal{X}_0)$ is shown in dark blue (dark shading). The hyperrectangular over-approximation of $R^F(1, \mathcal{X}_0)$, which is computed from the embedding system (24) as described in Proposition 3, is shown in light blue (light shading). (b) Visualization of the bounding procedure from Proposition 3. The trajectory of (24) which yields Figure 1a is plotted, where Φ^e is projected to the $x_1, \widehat{x}_1$ plane. The southeast cones corresponding to $\mathcal{X}_0$ and the hyperrectangular over-approximation of $R^F(1, \mathcal{X}_0)$ are shown in orange (medium shading) and light blue (light shading), respectively. (c) Approximating $R^F(0.25, \mathcal{X}_0)$. $\mathcal{X}_0$ is shown in orange (medium shading). $R^F(1/4, \mathcal{X}_0)$ is shown in dark blue (dark shading) with over-approximations derived from d and d' shown in light blue (light inner shading) and red (light outer shading), respectively.

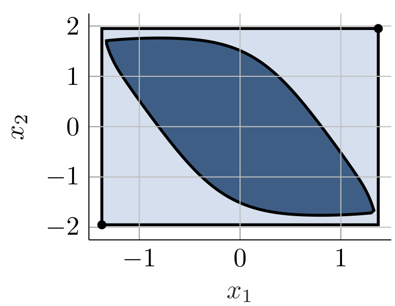

Fig. 2: Computing attractive sets of (31) in Example 7. The set $\mathcal{X}_{\text{eq}}$ from (33) is lightly shaded and is guaranteed by Corollary 4 to be an over-approximation of the darkly shaded region, which is the smallest globally attractive set for (31).

monotone with decomposition function

$$d(x, w, \widehat{x}, \widehat{w}) = \begin{bmatrix} -x_1 - x_1^3 - \widehat{x}_2 - \widehat{w} \\ -x_2 - x_2^3 + x_1 + w^3 \end{bmatrix}. \tag{32}$$

Additionally, $e(x_{\text{eq}}, \widehat{x}_{\text{eq}}) = 0$ for

$$x_{\text{eq}} = (-1.37, -1.95), \qquad \widehat{x}_{\text{eq}} = (1.37, 1.95), \tag{33}$$

and $(x_{\text{eq}}, \widehat{x}_{\text{eq}}) \in \mathcal{T}$. Therefore, from Theorem 2, $\mathcal{X}_{\text{eq}} = [x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is robustly forward invariant for (31). Moreover, it can be verified that $(x_{\text{eq}}, \widehat{x}_{\text{eq}})$ is globally asymptotically stable for (24); hence, from Corollary 4, it follows that $\mathcal{X}_{\text{eq}}$ is globally attractive for (31). The set $\mathcal{X}_{\text{eq}}$ is shown in Figure 2 along with a smallest attractive set determined via exhaustive simulation. ■

VI. Discrete-time Mixed Monotonicity

In this section, we collect the analogous definitions and results from Sections III and IV for discrete-time systems. Thus, we consider systems of the form

$$x^+ = F(x, w) \tag{34}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $w \in \mathcal{W} \subseteq \mathbb{R}^m$ and x^+ denotes the state at the next time step. We again assume that $\mathcal{X}$ is an extended hyperrectangle and that $\mathcal{W} = [\underline{w}, \overline{w}]$ is a hyperrectangle for some $\underline{w}, \overline{w} \in \mathbb{R}^m$. We assume that $F : \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$ is continuous. We do not assume that F maps only to $\mathcal{X}$, but the results below only hold as long as the state trajectory remains in $\mathcal{X}$. Throughout, we will introduce new definitions as necessary that replace their continuous-time counterparts above, such as the following.

Definition 4. A set $A \subseteq \mathcal{X}$ is *robustly forward invariant* for (34) if $F(x, w) \in A$ for all $x \in A$ and all $w \in \mathcal{W}$. When F does not depend on w, A is said to be *forward invariant*. ■

Mixed monotonicity in discrete-time is defined as follows, which can be considered the discrete-time analog to the Kamke conditions in Fact 1.

Definition 5. Given a continuous function $D : \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$, the system (34) is *mixed monotone with respect to D* if

- $F(x, w) = D(x, w, x, w)$ for all $x \in \mathcal{X}$ and for all $w \in \mathcal{W}$,
- $D(x, w, \widehat{x}, \widehat{w}) \leq D(y, v, \widehat{x}, \widehat{w})$ for all $x, y, \widehat{x} \in \mathcal{X}$ such that $x \leq y$ and for all $w, v, \widehat{w} \in \mathcal{W}$ such that $w \leq v$.
- $D(x, w, \widehat{y}, \widehat{v}) \leq D(x, w, \widehat{x}, \widehat{w})$ for all $x, \widehat{x}, \widehat{y} \in \mathcal{X}$ such that $\widehat{x} \leq \widehat{y}$ and for all $w, \widehat{w}, \widehat{v} \in \mathcal{W}$ such that $\widehat{w} \leq \widehat{v}$.

■

The system (34) is *monotone* if $F(x, w) \leq F(\widehat{x}, \widehat{w})$ whenever $x \leq \widehat{x}$ and $w \leq \widehat{w}$. Monotone discrete-time systems are also mixed monotone.

Example 8. The system (34) is monotone if and only if it is mixed monotone with respect to the particular decomposition function $D(x, w, \widehat{x}, \widehat{w}) = F(x, w)$ for all $x, \widehat{x} \in X$ and all $w, \widehat{w} \in \mathcal{W}$. ■

As in continuous-time, there always exists a decomposition function for (34), as established in [29].

Theorem 3 ([29, Theorem 1])**.** *Any system of the form* (34) *is mixed monotone with respect to a decomposition function* D *satisfying*

$$D_i(x, w, \widehat{x}, \widehat{w}) = \begin{cases} \min\limits_{\substack{y\in[x,\widehat{x}]\\ z\in[w,\widehat{w}]}} F_i(y, z) & \text{if } x \le \widehat{x} \text{ and } w \le \widehat{w}, \\ \max\limits_{\substack{y\in[\widehat{x},x]\\ z\in[\widehat{w},w]}} F_i(y, z) & \text{if } \widehat{x} \le x \text{ and } \widehat{w} \le w. \end{cases} \tag{35}$$

The decomposition function characterized by (35) is the tightest decomposition function and produces tight reachable set approximations. In particular, in discrete-time, applying Proposition 5 below to compute reachable sets via (45) using the decomposition function characterized in Theorem 3 produces the tightest hyperrectangle that over-approximates the true reachable set over one time step. However, iterating the embedding dynamics beyond one time step generally results in hyperrectangular reachable set approximations that are no longer tight.

When F is Lipschitz, we have the following Special Case 2 analogous to Special Case 1. The only difference between Special Case 2 for discrete-time systems and Special Case 1 above for continuous-time systems is the additional constraint on the diagonal entries of $\partial F/\partial x$ in Special Case 2.

Special Case 2. Consider (34) and assume further that F is Lipschitz in x and w. If there exist $\underline{J}_x \in \overline{\mathbb{R}}^{n\times n}_{\le 0}$, $\overline{J}_x \in \overline{\mathbb{R}}^{n\times n}_{\ge 0}$, $\underline{J}_w \in \overline{\mathbb{R}}^{n\times m}_{\le 0}$, and $\overline{J}_w \in \overline{\mathbb{R}}^{n\times m}_{\ge 0}$ such that

- for all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$,
$$\frac{\partial F}{\partial x}(x, w) \in [\underline{J}_x, \overline{J}_x] \text{ and } \frac{\partial F}{\partial w}(x, w) \in [\underline{J}_w, \overline{J}_w] \tag{36}$$
whenever the derivative exists,
- for all i, j, $(\underline{J}_x)_{i,j} > -\infty$ or $(\overline{J}_x)_{i,j} < \infty$, and
- for all i, k, $(\underline{J}_w)_{i,k} > -\infty$ or $(\overline{J}_w)_{i,k} < \infty$

then (34) is mixed monotone and a decomposition function is constructed in the following way:

1) For all $i, j \in \{1, \ldots, n\}$ and all $k \in \{1, \ldots, m\}$ choose $\delta_{i,j}, \epsilon_{i,k} \in \{0, 1\}$ such that
$$\begin{aligned} \delta_{i,j} = 0 &\quad \text{implies} \quad (\underline{J}_x)_{i,j} \neq -\infty, \\ \delta_{i,j} = 1 &\quad \text{implies} \quad (\overline{J}_x)_{i,j} \neq \infty, \\ \epsilon_{i,k} = 0 &\quad \text{implies} \quad (\underline{J}_w)_{i,k} \neq -\infty, \\ \epsilon_{i,k} = 1 &\quad \text{implies} \quad (\overline{J}_w)_{i,k} \neq \infty. \end{aligned} \tag{37}$$
Note that such a choice exists by hypothesis.
2) For all $i \in \{1, \ldots, n\}$, define $\xi^i, \alpha^i \in \mathbb{R}^n$ and $\pi^i, \beta^i \in \mathbb{R}^m$ element-wise according to
$$(\xi^i_j, \alpha^i_j) = \begin{cases} (x_j, -(\underline{J}_x)_{i,j}) & \text{if } \delta_{i,j} = 0, \\ (\widehat{x}_j, (\overline{J}_x)_{i,j}) & \text{if } \delta_{i,j} = 1, \end{cases} \tag{38}$$
$$(\pi^i_k, \beta^i_k) = \begin{cases} (w_k, -(\underline{J}_w)_{i,k}) & \text{if } \epsilon_{i,k} = 0, \\ (\widehat{w}_k, (\overline{J}_w)_{i,k}) & \text{if } \epsilon_{i,k} = 1. \end{cases} \tag{39}$$
3) Lastly, define the i^{th} element of the decomposition function D according to
$$\begin{aligned} D_i(x, w, \widehat{x}, \widehat{w}) = F_i(\xi^i, \pi^i) + (\alpha^i)^T(x - \widehat{x}) \\ + (\beta^i)^T(w - \widehat{w}), \end{aligned} \tag{40}$$
which is always well-defined on $\mathcal{X}\times\mathcal{W}\times\mathcal{X}\times\mathcal{W}$ since $\mathcal{X}$ and $\mathcal{W}$ are assumed to be extended hyperrectangles. ■

Linear systems always satisfy the conditions of Special Case 2.

Example 9. Any linear system $x^+ = Ax + Bw$ is mixed monotone with respect to the decomposition function $D(x, w, \widehat{x}, \widehat{w}) = \max\{0, A\}x + \max\{0, B\}w + \min\{0, A\}\widehat{x} + \min\{0, B\}\widehat{w}$ where the min and max operators are understood to be elementwise. This is the decomposition function given by the procedure of Special Case 2 with the choice $\delta_{i,j} = 0$ if and only if $A_{i,j} \ge 0$ and $\epsilon_{i,k} = 0$ if and only if $B_{i,k} \ge 0$ and taking $\underline{J}_x = \min\{0, A\}$, $\overline{J}_x = \max\{0, A\}$, $\underline{J}_w = \min\{0, B\}$, and $\overline{J}_w = \max\{0, B\}$. ■

Example 10. A common method of obtaining a discrete-time system is by sampling a continuous-time system. In particular, consider (1) sampled every T time units to obtain the discrete-time system $x^+ = F(x, w) := \phi(T, x, w)$ where the notation $\phi(T, x, w)$ is used to indicate that the disturbance is held constant with value w over the sampling period T. Mixed monotonicity of the sampled system then depends on properties of the flow map ϕ and is qualitatively different than continuous-time mixed monotonicity described above. For example, Special Case 2 amounts to conditions on $\frac{\partial\phi}{\partial x}(T, x, w)$, the sensitivity matrix for the system (1). See [25] for further details. ■

Consider system (34) that is mixed monotone with respect to decomposition function D. Completely analogously to Section IV, we define two embedding systems. We consider

$$\begin{bmatrix} x^+ \\ \widehat{x}^+ \end{bmatrix} = \mathcal{E}(x, w, \widehat{x}, \widehat{w}) := \begin{bmatrix} D(x, w, \widehat{x}, \widehat{w}) \\ D(\widehat{x}, \widehat{w}, x, w) \end{bmatrix} \tag{41}$$

and

$$\begin{bmatrix} x^+ \\ \widehat{x}^+ \end{bmatrix} = E(x, \widehat{x}) := \begin{bmatrix} D(x, \underline{w}, \widehat{x}, \overline{w}) \\ D(\widehat{x}, \overline{w}, x, \underline{w}) \end{bmatrix}. \tag{42}$$

For $a = (x, \widehat{x}) \in \mathcal{X} \times \mathcal{X}$ and $b = (w, \widehat{w}) \in \mathcal{W} \times \mathcal{W}$, we sometimes write $\mathcal{E}(a, b)$ to mean $\mathcal{E}(x, w, \widehat{x}, \widehat{w})$, and similarly write $E(a)$ for $E(x, \widehat{x})$. The k-fold composition of E is denoted $E^k(\cdot)$.

The following proposition recalls the canonical monotonicity properties of the embedding systems above with respect to the southeast order.

Proposition 4. *Suppose* (34) *is mixed monotone with respect to D. For any $a, a' \in \mathcal{X} \times \mathcal{X}$ and $b, b' \in \mathcal{W} \times \mathcal{W}$, if $a \leq_{\text{SE}} a'$ and $b \leq_{\text{SE}} b'$, then*

$$\mathcal{E}(a, b) \leq_{\text{SE}} \mathcal{E}(a', b') \tag{43}$$

and

$$E(a) \leq_{\text{SE}} E(a'). \tag{44}$$

The next proposition establishes the fundamental reachability result that the one-step reachable set of the nondeterministic dynamics (34) for a hyperrectangular set of initial states is over-approximated by a hyperrectangle defined by the evaluation of the decomposition function at two extreme points.

Proposition 5 ([19, Theorem 1])**.** *Suppose* (34) *is mixed monotone with respect to D. Given an initial hyperrectangle of states $\mathcal{X}_0 = [\underline{x}, \overline{x}] \subseteq \mathcal{X}$ and $w_1, w_2 \in \mathcal{W}$ such that $w_1 \leq w_2$,*

$$F(x, w) \in [D(\underline{x}, w_1, \overline{x}, w_2), D(\overline{x}, w_2, \underline{x}, w_1)] \tag{45}$$

for all $x \in \mathcal{X}_0$ and all $w \in [w_1, w_2]$.

Definition 6. A set $A \subset \mathcal{X}$ is *attractive for* (1) *from $\mathcal{X}' \subset \mathcal{X}$* if for any $x[0] \in \mathcal{X}'$, any infinite sequence $\{w[k]\}_{k=0}^{\infty}$ with $w[k] \in \mathcal{W}$ for all k, and each relatively open neighborhood $\mathcal{X}_\epsilon \subset \mathcal{X}$ of A, there exists $K > 0$ such that $x[k] \in \mathcal{X}_\epsilon$ for all $k \geq K$ where $x[k]$ is the unique solution to (34) satisfying $x[k+1] = F(x[k], w[k])$ for all $k \geq 0$. If A is attractive from some relatively open neighborhood of A, we say A is *locally attractive* or just *attractive*, and when A is attractive from $\mathcal{X}$, we say A is *globally attractive*. ■

The next theorem is the discrete-time analog to Theorem 2. We include a proof since the exact form of the theorem has not appeared in the literature before, although its proof is entirely analogous to the proof of Theorem 2.

Theorem 4. *Suppose* (34) *is mixed monotone with respect to D. For any $a = (x, \widehat{x}) \in \mathcal{T}$ such that $a \leq_{\text{SE}} E(a)$, where $\mathcal{T}$ is the upper triangle as defined in* (27)*, the following hold:*

1) *For any integer $K \geq 0$, the set $[\![E^K(a)]\!] \subseteq \mathcal{X}$ is robustly forward invariant for* (34)*.*
2) $\lim_{k\to\infty} E^k(a) =: (x_{\text{eq}}, \widehat{x}_{\text{eq}})$ *exists and* $E(x_{\text{eq}}, \widehat{x}_{\text{eq}}) = (x_{\text{eq}}, \widehat{x}_{\text{eq}})$, i.e., $(x_{\text{eq}}, \widehat{x}_{\text{eq}})$ *is an equilibrium for* (42)*.*
3) *The set $[x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is robustly forward invariant and attractive from $[\![a]\!] \subseteq \mathcal{X}$.*

Proof. For part 1, let $a = (x, \widehat{x})$ satisfy $a \in \mathcal{T}$, *i.e.*, $x \leq \widehat{x}$, and suppose $a \leq_{\text{SE}} E(a)$. Then, from Proposition 4, $E^k(a) \leq_{\text{SE}} E^{k+1}(a)$ for all $k \geq 0$. Choose $K \geq 0$ and let $b = E^K(a)$. Also from Proposition 4, $b \leq_{\text{SE}} E(b)$, or, equivalently, $[\![E(b)]\!] \subseteq [\![b]\!]$. From Proposition 5, $F(x, w) \in [\![E(b)]\!]$ for all $x \in [\![b]\!]$ and $w \in \mathcal{W}$, and thus, $[\![b]\!]$ is robustly forward invariant for (34).

For part 2, we first claim that for all $a \in \mathcal{T}$, $E(a) \in \mathcal{T}$ whenever $E(a) \in \mathcal{X}$, a result analogous to Corollary 2. To establish the claim, let $(x_0, \widehat{x}_0) \in \mathcal{T}$ and note that $(x_0, \widehat{x}_0) \leq_{\text{SE}} (\widehat{x}_0, x_0)$. Let $b = (\underline{w}, \overline{w})$ and $b' = (\overline{w}, \underline{w})$ so that also $b \leq_{\text{SE}} b'$. Then $(x_1, \widehat{x}_1) := E(a) = \mathcal{E}(a, b) \leq_{\text{SE}} \mathcal{E}(a', b') = (\widehat{x}_1, x_1)$ where the inequality follows by Proposition 4 and the last equality by the definition of $\mathcal{E}$ in (41). Thus, $(x_1, \widehat{x}_1) \leq_{\text{SE}} (\widehat{x}_1, x_1)$, *i.e.*, $x_1 \leq \widehat{x}_1$, and the claim is proved. Now, as in part 1, $E^k(a) \leq_{\text{SE}} E^{k+1}(a)$ so that $E^k(a)$ is an increasing sequence with respect to the order $\leq_{\text{SE}}$. Combining these observations, $E^k(a) \in \{(x, \widehat{x}) \in \mathcal{X} \times \mathcal{X} \mid x_0 \leq x \leq \widehat{x} \leq \widehat{x}_0\}$ for all $k \geq 0$ where $(x_0, \widehat{x}_0)$ is defined as $(x_0, \widehat{x}_0) = a$. Since $E^k(a)$ is increasing in k and bounded, the limit $\lim_{k\to\infty} E^k(a) =: (x_{\text{eq}}, \widehat{x}_{\text{eq}})$ exists and satisfies $E(x_{\text{eq}}, \widehat{x}_{\text{eq}}) = (x_{\text{eq}}, \widehat{x}_{\text{eq}})$.

For part 3, choose $x \in [\![a]\!]$ and choose any sequence $\{w[k]\}_{k=0}^{\infty}$ with $w[k] \in \mathcal{W}$ for all $k \geq 0$, and let $x[k]$ satisfy the state update $x[k+1] = F(x[k], w[k])$ for all $k \geq 0$. It follows that $x[k] \in [\![E^k(a)]\!]$ for all $k \geq 0$. Choose a relatively open neighborhood $\mathcal{X}_\epsilon$ of $[x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ and a relatively open ball $B \subset \mathcal{X} \times \mathcal{X}$ such that $(x_{\text{eq}}, \widehat{x}_{\text{eq}}) \in B \subset \mathcal{X}_\epsilon \times \mathcal{X}_\epsilon$. From part 2 above, there must exist a $K \geq 0$ such that $E^K(a) \in B$ and therefore $x[K] \in \mathcal{X}_\epsilon$. From part 1, $[\![E^K(a)]\!]$ is robustly forward invariant for (34) and therefore $x[k] \in \mathcal{X}_\epsilon$ for all $k \geq K$. Thus, $[x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is attractive for (34) from $[\![a]\!]$. The fact that $[x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is robustly forward invariant follows immediately from part 1. □

Analogous to Corollary 4, the following corollary specializes Theorem 4 to asymptotically stable equilibria of (34).

Corollary 5. *Suppose* (34) *is mixed monotone with respect to D. If $(x_{\text{eq}}, \widehat{x}_{\text{eq}}) \in \mathcal{T}$ is an asymptotically stable equilibrium for* (42) *with a basin of attraction $\mathcal{C} \subseteq \mathcal{X} \times \mathcal{X}$, then $[x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is robustly forward invariant for* (34) *and attractive from any set $[\![a]\!]$ such that $a \in \mathcal{C} \cap \mathcal{T}$. In particular, if $\mathcal{C} \supseteq \mathcal{T}$, then $[x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is globally attractive and robustly forward invariant for* (34)*.*

VII. Case Studies

A. Generalized Lotka-Volterra Equations

Consider the generalized Lotka-Volterra equations subject to disturbance given by

$$\dot{x}_i = x_i \left(b_i + w_i + \sum_{j=1}^{n} c_{ij} x_j \right) \quad \text{for } i = 1, \ldots n \tag{46}$$

with $\mathcal{X} = \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\mathcal{W} = [\underline{w}, \overline{w}]$ for some $\underline{w}, \overline{w} \in \mathbb{R}^n$ where each b_i and c_{ij} are constants with no a priori restriction on their sign. Each state x_i is the population of some species in an ecological system. This system is mixed monotone with respect to the decomposition function

$$\begin{aligned} d_i(x, \widehat{x}, w, \widehat{w}) &= x_i(b_i + w_i + c_{ii} x_i) \\ &+ x_i \left(\sum_{j \neq i} \max\{0, c_{ij}\} x_j + \sum_{j \neq i} \min\{0, c_{ij}\} \widehat{x}_j \right). \end{aligned} \tag{47}$$

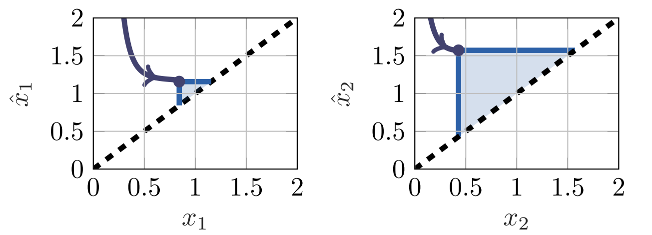

Fig. 3: A sample trajectory of the embedding system (24) constructed from the decomposition function (49) for the system (48) with $\mathcal{W} = [(-0.1, -0.1), (0.1, 0.1)] \subset \mathbb{R}^2$. The initial condition $a = (x_1, x_2, \widehat{x}_1, \widehat{x}_2) = (0.3, 0.3, 2, 2)$ converges to the equilibrium $(x_{\text{eq}}, \widehat{x}_{\text{eq}}) = (0.843, 0.429, 1.157, 1.571)$. Therefore, the hyperrectangle $[(0.843, 0.429), (1.157, 1.571)] \subset \mathbb{R}^2$ is robustly forward invariant and attractive from $[\![a]\!] = [(0.3, 0.3), (2, 2)] \subset \mathbb{R}^2$ for the dynamics (48) by Theorem 2.

The essential observation that Lotka-Volterra models are decomposable in this way is first noted in [17], although in [17], a change of coordinates from x to $\ln(x)$ is used and no disturbance is present. Note that, in general, (46) does not satisfy the hypotheses of Special Case 1.

As an example, consider

$$\begin{aligned} \dot{x}_1 &= x_1(1.1 + w_1 - x_1 - 0.1x_2) \\ \dot{x}_2 &= x_2(4 + w_2 - 3x_1 - x_2). \end{aligned} \tag{48}$$

This example, without the disturbances w_1 and w_2, appears in [35]. Without disturbance, $(x_1, x_2) = (1, 1)$ is an asymptotically stable equilibrium. The decomposition function constructed according to (47) is given by

$$d(x, \widehat{x}, w, \widehat{w}) = \begin{bmatrix} x_1(1.1 + w_1 - x_1 - 0.1\widehat{x}_2) \\ x_2(4 + w_2 - 3\widehat{x}_1 - x_2) \end{bmatrix}. \tag{49}$$

Take $\underline{w}_1 = \underline{w}_2 = -0.1$ and $\overline{w}_1 = \overline{w}_2 = 0.1$. It can be verified that the embedding system (24) constructed from (49) has an equilibrium $(x_{\text{eq}}, \widehat{x}_{\text{eq}}) = (0.843, 0.429, 1.157, 1.571)$ that is asymptotically stable. Thus, from Theorem 2, the rectangle $[(0.843, 0.429), (1.157, 1.571)]$ is robustly forward invariant and attractive from $[\![a]\!]$ for any $a \in \mathbb{R}^{2n}$ in the basin of attraction of $(x_{\text{eq}}, \widehat{x}_{\text{eq}})$ with $a \leq_{\text{SE}} (x_{\text{eq}}, \widehat{x}_{\text{eq}})$. Figure 3 shows a sample trajectory of (24) with the initial condition $a = (0.3, 0.3, 2, 2)$ within the basin of attraction of $(x_{\text{eq}}, \widehat{x}_{\text{eq}})$.

B. Discrete-time Population Dynamics

A discrete-time analog to the generalized Lotka-Volterra equations is given by the matrix population model of the form [36]

$$x^+ = A(x)x + w \tag{50}$$

with $\mathcal{X} = \{x : \mathbb{R}^n \mid x \geq 0\}$ and $\mathcal{W} = [\underline{w}, \overline{w}]$ for some $\underline{w}, \overline{w} \in \mathbb{R}^n$ with $\underline{w} \geq 0$ where each state x_i is again the population of some species in an ecological system, but now the time variable is discrete and represents generations, and w is a per-generation restocking rate. If $A(x) \geq 0$ for all x, and $x \leq \widehat{x}$ implies $A(\widehat{x}) \leq A(x)$, then the system is mixed monotone with respect to the decomposition function $D(x, w, \widehat{x}, \widehat{w}) = A(\widehat{x})x + w$, as first observed in [13]. In general, this decomposition is not the tightest decomposition function constructed in Theorem 3, nor does (50) generally satisfy the conditions of Special Case 2.

For example, consider the matrix population model of the flour beetle *Tribolium casteneum* [13], [37] given by (50) with $x \in \mathbb{R}^3$ and

$$A(x) = \begin{bmatrix} 0 & 0 & de^{-ax_1 - bx_2} \\ p & 0 & 0 \\ 0 & qe^{-cx_3} & r \end{bmatrix} \tag{51}$$

where $p, q, r \in (0, 1]$ and $a, b, c, d > 0$ for which $A(x) \geq 0$ for all x and $x \leq \widehat{x}$ implies $A(\widehat{x}) \leq A(x)$, and we do not consider restocking so that $\mathcal{W} = \{0\}$. The states x_1, x_2, and x_3 denote the larvae, pupae, and adult populations of the flour beetle. Values for parameters are taken to be $a = 0.005$, $b = 0.011$, $c = 0.004$, $d = 7.88$, $p = 0.8390$, $q = 1$, and $r = 0.75$ [37]. If initially $x_i[0] \in [5, 10]$ for $i = 1, 2, 3$, then, applying Proposition 5, the populations at the second subsequent generation satisfy $x_1[2] \in [38.7, 104.9]$, $x_2[2] \in [28.2, 61.0]$, and $x_3[2] \in [10.3, 21.1]$.

C. Vehicle Platoon

Consider a platoon of N vehicles with velocity dynamics

$$\dot{v}_i = -k_i v_i + c_i + u_i + w_i \tag{52}$$

where $v_i \in \mathbb{R}$ is the velocity of vehicle $i \in \{1, \ldots, N\}$, c_i is the fixed nominal velocity of vehicle i, $k_i > 0$ is a fixed damping constant, u_i is a control input based on relative displacements as defined below, and w_i is a disturbance. Let p_i denote the position of vehicle i, *i.e.*, p_i satisfies $\dot{p}_i = v_i$. We assume relative displacements are available via an undirected communication graph with L edges. After arbitrarily assigning an orientation (*i.e.*, head and tail) to each edge, this graph is represented by an $N \times L$ incidence matrix $\mathcal{G}$ given elementwise by

$$\mathcal{G}_{i,\ell} = \begin{cases} 1 & \text{if vehicle } i \text{ is the head of edge } \ell \\ -1 & \text{if vehicle } i \text{ is the tail of edge } \ell \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

The vector of L relative displacements is then given by $z = \mathcal{G}^T p$. We consider a control policy as proposed in, *e.g.*, [38, Chapter 4], given by

$$u = -\mathcal{G}h(z) \tag{54}$$

where $h(z) = \begin{bmatrix} h_1(z_1) & \ldots & h_M(z_M) \end{bmatrix}^T$ and each $h_i : \mathbb{R} \to \mathbb{R}$ is strictly increasing. The closed loop system is then defined by (52)–(54) with state $x = \begin{bmatrix} v^T & z^T \end{bmatrix}^T$; denote this system as $\dot{x} = f(x, w)$. Then the off-diagonal entries of $\partial f / \partial x$ and all entries of $\partial f / \partial w$ do not change sign, and therefore, as first observed in [39], the system satisfies the hypotheses of Special Case 1 and is mixed monotone with respect to the decomposition function

$$d(x,\widehat{x},w,\widehat{w}) = d\left(\begin{bmatrix} v \\ z \end{bmatrix}, \begin{bmatrix} \widehat{v} \\ \widehat{z} \end{bmatrix}, w, \widehat{w}\right) \tag{55}$$

$$= \begin{bmatrix} -Kv - \min\{0,\mathcal{G}\}h(z) - \max\{0,\mathcal{G}\}h(\widehat{z}) + c + w \\ \max\{0,\mathcal{G}^T\}v + \min\{0,\mathcal{G}^T\}\widehat{v} \end{bmatrix} \tag{56}$$

where $K = \mathrm{diag}\{k_1,\dots,k_N\}$ and the $\min$ and $\max$ operations are understood to be elementwise.

As an example, take $N = 3$, $L = 2$, $h_i(s) = \tanh(s)$, $c_1 = c_2 = 0$, $k_1 = k_2 = -1$ and

$$\mathcal{G} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{57}$$

so that $z_1 = p_2 - p_1$ and $z_2 = p_3 - p_2$. Let $w \in [\underline{w}, \overline{w}]$ and take $\underline{w}_1 = \underline{w}_2 = -0.1$ and $\overline{w}_1 = \overline{w}_2 = 0.1$. Suppose initially $-0.1 \leq v_i(0) \leq 0.1$ for $i = 1,2,3$ and $z_1(0) = z_2(0) = 1$ and that the vehicle positions have been normalized so that a collision corresponds to $z_1 \leq -1$ or $z_2 \leq -1$. By simulating a trajectory of (24) to compute an over-approximation of the reachable set as guaranteed by Proposition 3, we verify that the system remains collision-free for $0 \leq t \leq 2.25$.

VIII. Acknowledgments

The author thanks Matthew Abate and Maxence Dutreix for providing feedback and discussion and also for providing Examples 2, 5, 6 and 7.

References

[1] P. Tabuada, *Verification and control of hybrid systems: A symbolic approach*. Springer, 2009.

[2] C. Belta, B. Yordanov, and E. A. Gol, *Formal Methods for Discrete-Time Dynamical Systems*, vol. 89. Springer, 2017.

[3] A. A. Kurzhanskiy and P. Varaiya, "Computation of reach sets for dynamical systems," in *The Control Systems Handbook, Chapter 29*, pp. 709–740, CRC Press, 2018.

[4] A. Girard, "Reachability of uncertain linear systems using zonotopes," in *International Workshop on Hybrid Systems: Computation and Control*, pp. 291–305, Springer, 2005.

[5] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied interval analysis*. Springer, 2001.

[6] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of linear systems with uncertain parameters and inputs," in *IEEE Conference on Decision and Control*, pp. 726–732, 2007.

[7] C. Fan, B. Qi, S. Mitra, M. Viswanathan, and P. S. Duggirala, "Automatic reachability analysis for nonlinear hybrid models with C2E2," in *International Conference on Computer Aided Verification*, pp. 531–538, Springer, 2016.

[8] C. Fan, J. Kapinski, X. Jin, and S. Mitra, "Simulation-driven reachability using matrix measures," *ACM Transactions on Embedded Computing Systems*, vol. 17, no. 1, pp. 1–28, 2017.

[9] J. Maidens and M. Arcak, "Reachability analysis of nonlinear systems using matrix measures," *IEEE Transactions on Automatic Control*, vol. 60, no. 1, pp. 265–270, 2014.

[10] D. Angeli and E. Sontag, "Monotone control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, pp. 1684–1698, 2003.

[11] H. L. Smith, *Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems*. American Mathematical Society, 1995.

[12] M. W. Hirsch, "Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere," *SIAM Journal on Mathematical Analysis*, vol. 16, no. 3, pp. 423–439, 1985.

[13] H. Smith, "The discrete dynamics of monotonically decomposable maps," *Journal of Mathematical Biology*, vol. 53, no. 4, pp. 747–758, 2006.

[14] J.-L. Gouzé and K. Hadeler, "Monotone flows and order intervals," *Nonlinear World*, vol. 1, pp. 23–34, 1994.

[15] G. Enciso, H. Smith, and E. Sontag, "Nonmonotone systems decomposable into monotone systems with negative feedback," *Journal of Differential Equations*, vol. 224, no. 1, pp. 205–227, 2006.

[16] H. Smith, "Global stability for mixed monotone systems," *Journal of Difference Equations and Applications*, vol. 14, no. 10-11, pp. 1159–1164, 2008.

[17] J.-L. Gouzé, "A criterion of global convergence to equilibrium for differential systems. application to Lotka-Volterra systems," *[Research Report] RR-0894, INRIA*, 1988.

[18] D. Angeli, G. A. Enciso, and E. D. Sontag, "A small-gain result for orthant-monotone systems under mixed feedback," *Systems & Control Letters*, vol. 68, pp. 9–19, 2014.

[19] S. Coogan and M. Arcak, "Efficient finite abstraction of mixed monotone systems," in *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, pp. 58–67, 2015.

[20] T. Moor and J. Raisch, "Abstraction based supervisory controller synthesis for high order monotone continuous systems," in *Modelling, Analysis, and Design of Hybrid Systems*, pp. 247–265, Springer, 2002.

[21] P.-J. Meyer, A. Girard, and E. Witrant, "Safety control with performance guarantees of cooperative systems using compositional abstractions," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 317–322, 2015.

[22] A. Saoud, E. Ivanova, and A. Girard, "Efficient synthesis for monotone transition systems and directed safety specifications," in *IEEE Conference on Decision and Control*, pp. 6255–6260, 2019.

[23] S. Coogan, E. A. Gol, M. Arcak, and C. Belta, "Traffic network control from temporal logic specifications," *IEEE Transactions on Control of Network Systems*, vol. 3, pp. 162–172, June 2016.

[24] S. Coogan and M. Arcak, "Stability of traffic flow networks with a polytree topology," *Automatica*, vol. 66, pp. 246–253, April 2016.

[25] P.-J. Meyer, S. Coogan, and M. Arcak, "Sampled-data reachability analysis using sensitivity and mixed-monotonicity," *IEEE Control Systems Letters*, pp. 761–766, 2018.

[26] P. Meyer and D. V. Dimarogonas, "Hierarchical decomposition of LTL synthesis problem for nonlinear control systems," *IEEE Transactions on Automatic Control*, vol. 64, pp. 4676–4683, Nov 2019.

[27] M. Dutreix and S. Coogan, "Specification-guided verification and abstraction refinement of mixed-monotone stochastic systems," *IEEE Transactions on Automatic Control*, to appear, 2021.

[28] L. Yang, O. Mickelin, and N. Ozay, "On sufficient conditions for mixed monotonicity," *IEEE Transactions on Automatic Control*, vol. 64, pp. 5080–5085, Dec 2019.

[29] L. Yang and N. Ozay, "Tight decomposition functions for mixed monotonicity," in *IEEE Conference on Decision and Control*, pp. 5318–5322, Dec 2019.

[30] M. Abate, M. Dutreix, and S. Coogan, "Tight decomposition functions for continuous-time mixed-monotone systems with disturbances," *IEEE Control Systems Letters*, vol. 5, pp. 139–144, Jan. 2021.

[31] P.-J. Meyer, A. Devonport, and M. Arcak, "TIRA: Toolbox for interval reachability analysis," in *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, HSCC '19, pp. 224–229, Association for Computing Machinery, 2019.

[32] M. Abate and S. Coogan, "Computing robustly forward invariant sets for mixed-monotone systems," in *IEEE Conference on Decision and Control*, 2020.

[33] S. Coogan, M. Arcak, and A. A. Kurzhanskiy, "Mixed monotonicity of partial first-in-first-out traffic flow models," in *Conference on Decision and Control*, pp. 7611–7616, Dec 2016.

[34] D. Angeli and E. D. Sontag, "Behavior of responses of monotone and sign-definite systems," *Mathematical Systems Theory*, pp. 51–64, 2013.

[35] B. S. Goh, "Global stability in many-species systems," *The American Naturalist*, vol. 111, no. 977, pp. 135–143, 1977.

[36] J. M. Cushing, *An introduction to structured population dynamics*. SIAM, 1998.

[37] R. Costantino, J. Cushing, B. Dennis, and R. A. Desharnais, "Experimentally induced transitions in the dynamic behaviour of insect populations," *Nature*, vol. 375, no. 6528, pp. 227–230, 1995.

[38] M. Arcak, C. Meissen, and A. Packard, *Networks of Dissipative Systems*. Springer, 2016.

[39] M. Abate and S. Coogan, "Enforcing safety at runtime for systems with disturbances," in *IEEE Conference on Decision and Control*, 2020.