The Price of Anarchy for Transportation Networks with Mixed Autonomy*

Daniel A. Lazar[1], Samuel Coogan[2] and Ramtin Pedarsani[3]

Abstract— **We study routing behavior in transportation networks with mixed autonomy, that is, networks in which a fraction of the vehicles on each road are equipped with autonomous capabilities such as adaptive cruise control that enable reduced headways and increased road capacity. Motivated by capacity models developed for such roads with mixed autonomy, we consider transportation networks in which the delay on each road or link is an affine function of two quantities: the number of vehicles with autonomous capabilities on the link and the number of regular vehicles on the link.**

We particularly study the price of anarchy for such networks, that is, the ratio of the total delay experienced by selfish routing to the socially optimal routing policy. Unlike the case when all vehicles are of the same type, for which the price of anarchy is known to be bounded, we first show that the price of anarchy can be arbitrarily large for such mixed autonomous networks. Next, we define a notion of asymmetry corresponding to the maximum possible travel time improvement due to the presence of autonomous vehicles. We show that when the degree of asymmetry of all links in the network is bounded by a factor less than 4, the price of anarchy is bounded. We also bound the bicriteria, which is a bound on the cost of selfishly routing traffic compared to the cost of optimally routing additional traffic. These bounds depend on the degree of asymmetry and recover classical bounds on the price of anarchy and bicriteria in the case when no asymmetry exists. Further, we show with examples that these bound are tight in particular cases.

I. INTRODUCTION

Automobiles are increasingly equipped with autonomous and semi-autonomous technologies such as adaptive cruise control and automated lane-keeping. These technologies are often marketed to consumers as safety or convenience features, but it is apparent that increasing numbers of these *smart* vehicles will have dramatic impact on network-level mobility factors such as traffic congestion and travel times [1]. A primary mechanism whereby such autonomous capabilities can improve mobility is by enabling *platooning* of groups of smart vehicles along the roadway. A platoon consists of two or more vehicles which are able to automatically maintain short headways between them using, *e.g.*, adaptive cruise control (ACC), which allows a vehicle to use radar or LIDAR to automatically maintain a specified distance to the preceding vehicle, or cooperative adaptive cruise control (CACC) which augments ACC with vehicle-to-vehicle communication.

When all vehicles in the system are smart, platooning has the potential to increase network capacity by as much as three-fold [2]. Platooning can help to smooth traffic flow and avoid *shockwaves* of slowing vehicles [3], [4], [5], [6], [7], [8], and at signalized intersections, platoons can synchronously accelerate at green lights [2], [9]. However, in a *mixed* autonomy setting—where only a fraction of the vehicles are smart and the remainder are *regular*, human-driven vehicles—the benefits of platooning are less clear. On freeways, simulation results suggest that high penetration rates of smart vehicles are required to realize significant improvement in traffic flow [10], [11], [12], [13], [14], [15].

In our prior research, we developed an analytic model for the capacity of roads at signalized intersections with mixed autonomous traffic [16]. There, we consider a queue of vehicles at an intersection and suppose that smart vehicles platoon opportunistically, that is, if a smart vehicle queues behind another smart vehicle, they maintain a short headway along the road. We also proposed and studied a second model in which each smart vehicle maintains a short headway to the preceding vehicle, whether it is also smart or not. Such a scenario may be increasingly possible as adaptive cruise control with passive sensing continues to improve. Our capacity models describe the maximum possible flow rate of vehicles through an intersection as a nonlinear function of the *level of autonomy* of the road, that is, the fraction of smart vehicles on the road.

In this work, we use the capacity models based on one we developed in [16] in order to study routing behavior on road networks. We make the assumption that the additional travel time caused by congestion on a road is inversely related to capacity and proportional to the total number of vehicles on the road. Given a network of roads leading from origins to destinations, selfish vehicles will choose the route that minimizes total delay, achieving a Wardrop equilibrium [17], [18]. It has long been known that a Wardrop equilibrium may be suboptimal in the sense that a social planner is able to prescribe routes that achieve a lower total delay for all vehicles in the network. The ratio of the socially optimal delay to the worst possible Wardrop equilibrium is called the *price of anarchy* [19], [20]. For affine separable cost functions, when only one type of vehicle is present (*i.e.*, no smart vehicles), it is known that the price of anarchy cannot exceed 4/3 [21].

In a mixed autonomy setting, however, a social planner is

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[1]Daniel Lazar is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara dlazar@ece.ucsb.edu

[2]Samuel Coogan is with the School of Electrical and Computer Engineering and the School of Civil and Environmental Engineering, Georgia Institute of Technology sam.coogan@gatech.edu

[3]Ramtin Pedarsani is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara ramtin@ece.ucsb.edu

able to route smart vehicles differently than regular vehicles to maximize capacity. In this paper, we first show that this increased flexibility leads, remarkably, to an unbounded price of anarchy. Next, we make the assumption that the possible travel time improvement due to the presence of autonomous vehicles is bounded by a factor $k < 4$. We call this factor the *degree of asymmetry* of the network. Under this assumption, we prove that the price of anarchy cannot exceed $\frac{4}{4-k}$, which recovers the classical bound when $k = 1$, *i.e.*, the case when smart vehicles do not enable any improvement in travel time. We show via examples that this bound is tight for $k = 1$ and $k = 2$.

Next, we provide a bound on the cost of selfish routing relative to the cost of optimally routing additional traffic, called the bicriteria bound [21], [22]. We prove that traffic at a Wardrop Equilibrium will not exceed the cost of optimally routing $1 + \frac{k}{4}$ as much traffic of each type, where k is the degree of asymmetry in the network. We demonstrate by example that the bicriteria bound is tight for $k = 4$. Similar to the price of anarchy, when the asymmetry is unbounded we show that the bicriteria is unbounded as well. This runs counter to the case of single-type traffic where the bicriteria is bounded by 2 for any separable continuous and nondecreasing cost function in which the delay on a road depends only on the traffic on that road [21].

II. PREVIOUS WORKS

In this section, we address related models in the literature and highlight the difference between these and our model. Due to the breadth of the field, we give a limited overview of the literature on the price of anarchy – see [23] for a broader survey of literature related to Wardrop equilibria and [24] for a wider background on the price of anarchy in transportation networks. For definitions of the terms used in this section see Section III-C.

Roughgarden and Tardos [21] bound price of anarchy and bicriteria for separable monotonic cost functions, and Roughgarden [25] gives a more general method for determining price of anarchy in the separable case.

Chau and Sim [26] bound the price of anarchy for symmetric cost operators with convex social cost for both nonelastic and elastic demands. Perakis discusses nonseparable, asymmetric, nonlinear costs in [27], though only for monotone latencies *i.e.* satisfying the property

$$\langle c(z) - c(v), z - v \rangle \geq 0 \ , \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors.

Correa et. al [28] present a unified framework for deriving price of anarchy and bicriteria for nonseparable monotone functions. Sekar et. al [29] analyze the price of anarchy when users have different beliefs about the delay on a road, but experience the same actual cost, dictated by a monotone cost function.

Unlike these previous works, we present a price of anarchy and bicriteria bound for a class of *nonmonotone* and pairwise separable affine cost functions. We show that our bound simplifies to the classic bounds for affine monotone cost functions in [21], [26], and [28] when there is no asymmetry in how the vehicle types affect congestion.

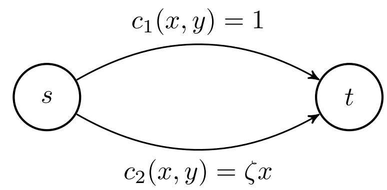

Fig. 1: Example of a road network with price of anarchy and bicriteria that grow unboundedly with ζ.

III. MOTIVATION AND MATHEMATICAL FORMULATION

In this section we motivate our cost function for traffic in mixed autonomy. In Section III-A, we show that the price of anarchy and bicriteria are unbounded for congestion games with affine cost functions in mixed autonomy, described in Section III-B. Prompted by our negative result, in Section III-C we describe a pairwise separable cost function that is parameterized by the degree of asymmetry, as well as a more general class of nonseparable cost function.

A. Motivation

We provide a brief example of unbounded price of anarchy and bicriteria for congestion games under mixed autonomy. This example is similar in design to Pigou's example, as in [21], [30], and [24].

Example 1: Consider the traffic network in Fig. 1, in which $\frac{1}{\zeta}$ unit of regular traffic and 1 unit of smart traffic need to travel from node s to node t, where $\zeta \geq 1$. The cost on road i, or the delay a car experiences from traveling on that road, is denoted $c_i(x, y)$.

The routing with all traffic on the bottom road is at Wardrop Equilibrium, with a price of $C^{EQ} = (\frac{1}{\zeta}\zeta)(\frac{1}{\zeta}+1) = \frac{1}{\zeta} + 1$. The optimal routing has the regular traffic on the top and smart traffic on the bottom with a cost of $C^{OPT} = \frac{1}{\zeta} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{\zeta}$. This results in a price of anarchy of $\zeta + 1$.

For the bicriteria, consider a situation in which we have a mass of $\frac{a}{\zeta}$ units of smart cars and a units of regular cars to route. We want to find the a that corresponds to, under optimal routing, a cost equaling that of C^{EQ} above. The optimal routing will have cost $\frac{a}{\zeta}$, which equals $\frac{1}{\zeta} + 1$ when $a = \zeta + 1$.

Here we see that both the price of anarchy and the bicriteria bound grow unboundedly with ζ. Due to this result, we state the following proposition:

Proposition 1: The price of anarchy and bicriteria are unbounded for networks of mixed autonomy with pairwise separable affine functions.

Because of this negative result, to provide a bounded price of anarchy and bicriteria in mixed autonomy, we develop a class of cost functions with bounded asymmetry.

B. Affine Congestion Game Overview

Consider a network of n roads, with m origin-destination pairs, each with an associated mass of regular vehicles of volume α_i and mass of smart vehicles of volume β_i. Since we are considering a nonatomic congestion game, each user controls an infinitesimally small portion of that mass. We denote $\mathcal{X}$ as the set of feasible strategies which result in the entirety of each mass being routed from its origin to its destination, without violating conservation of flow (see [31] for a more detailed explanation).

The vector of all flows on the n roads is denoted by

$$z = \begin{bmatrix} x_1 & y_1 & x_2 & y_2 & \dots & x_n & y_n \end{bmatrix}^T ,$$

where x_i and y_i represent the mass of regular and smart vehicles, respectively, on road i. In this paper, we consider affine cost functions, meaning the cost on the roads resulting from a routing $z \in \mathcal{X}$ can be written as

$$c(z) = Az + b ,$$

where $A \in \mathbb{R}^{2n}_{\geq 0}$ and $b = \begin{bmatrix} b_1 & b_1 & b_2 & b_2 & \dots & b_n & b_n \end{bmatrix}^T$. This yields social cost

$$C(z) = \langle c(z), z \rangle = (Az + b)^T z ,$$

and the social cost at optimal routing is then $C^{OPT} = \inf_{z \in \mathcal{X}} C(z)$. Vector b contains the constant terms; matrix A is the Jacobian of the road cost operator, and is not in general positive semidefinite, so the optimization is not convex.

C. Separability and Monotonicity

Having described the basic structure of the congestion game with affine costs, we describe the separability and monotonicity of our model. To do so, we define three notions of separability.

Definition 1: A cost function $c(z) = Az + b$ is *separable* if A is a diagonal matrix.

Definition 2: A cost function $c(z) = Az + b$ is *pairwise separable* if A is a blockwise diagonal matrix with 2x2 blocks.

Definition 3: A cost function is *nonseparable* if it is neither separable nor pairwise separable.

It is clear that separable costs do not model mixed autonomy if regular and smart cars affect delay differently but experience it identically. The slightly more general class of pairwise separable costs does provide a useful model, which we motivate as follow, using a capacity model similar to those in [9], [32], [16].

Consider a road of length d with regular car flow x and smart car flow y. Let ℓ be the (uniform) vehicle length, h_s the headway in front of a smart car, h_l the headway in front of a regular car, and autonomy level $\alpha = \frac{y}{x+y}$. We consider the capacity on this road as the number of cars that can be packed onto the road: $g(\alpha) = \frac{d}{\alpha h_s + (1-\alpha) h_l + \ell}$. We propose a road cost function in which delay is an affine function of vehicles on the road: $c(x, y) = b + r\frac{x+y}{g(x,y)} = b + \frac{r}{\ell}(h_l + \ell)x + \frac{r}{\ell}(h_s + \ell)y$. Here b_i represents the time it takes to traverse a road in free-flow traffic and r_i determines how congestion scales as road utilization increases with respect to capacity.

To capture a notion of asymmetry in how the types of vehicles affect traffic, let $a \triangleq \frac{r_i(h_s+\ell)}{\ell}$ and $k \triangleq \frac{h_l+\ell}{h_s+\ell}$. Indexing these values to be on road i,

$$c_i(x_i, y_i) = b_i + k_i a_i x_i + a_i y_i \ . \tag{2}$$

This leads to a cost function of the following form:

$$\begin{aligned} c(z) &= Az + b \\ &= \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix} z + b \end{aligned}$$

where A is a block-diagonal matrix with blocks $A_i = \begin{bmatrix} k_i a_i & a_i \\ k_i a_i & a_i \end{bmatrix}$.

The parameter k_i allows us to represent the degree of asymmetry between the effect of regular and smart traffic on congestion on a specific road. Since in [16], [33], and [2], we see that vehicles not in a platoon require approximately 2.5 times more headway than vehicles in a platoon, we allow k_i to differ between roads, but generally expect it to be in the range $k_i \in [1, 4]$.

We find it useful to parameterize a class of cost functions by its maximum degree of asymmetry, as follows:

Definition 4: Let $\mathcal{C}_k$ denote the class of pairwise separable cost functions for which $\max(k_i, \frac{1}{k_i}) \leq k \ \forall i$ for some constant k. We call k the *maximum degree of asymmetry* of this class of cost functions.

In the more general model explored in Section IV-C, the delay on one road may depend on the flows on other roads. For example, if one road is fully congested, the roads feeding it will have additional delay. If this is the case, then (2) does not hold and the matrix A is not of block-diagonal form. In Section IV-C we provide bounds for this model under certain conditions.

Throughout this paper, we deal with cost functions that satisfy element-wise monotonicity, defined as follows:

Definition 5: A class of cost functions $\mathcal{C}$ is *elementwise monotone* if for all cost functions $c(v)$ drawn from $\mathcal{C}$, $\frac{\partial c_j}{\partial v_i}(v) \geq 0 \ \forall i, j$.

In other words, a cost function is element-wise monotonic if increasing any flow of vehicles will not decrease the delay on any road. This will be the case for a class of cost functions of the form $c(z) = Az + b$ for which A has only nonnegative entries. Note that this is different from the general notion of monotonicity described in Section II[1].

IV. BOUNDING THE PRICE OF ANARCHY

In this section we present bounds for the price of anarchy and bicriteria. We proceed along the lines proposed in [28],

[1]Our case is not in general monotone: consider a single road with $c(z) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} z$, with $x = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $y = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$. This results in $\langle c(x) - c(y), x - y \rangle = -1$.

reviewing that work in Section IV-A and highlighting the differences that arise for a nonmonotone cost function. We then derive our bounds for nonmonotone pairwise separable costs in Section IV-B, and give bounds for nonseparable costs in Section IV-C.

A. Preliminaries

Smith [34] shows that any flow z^{EQ} at Wardrop equilibrium – in which all users sharing an origin and destination use paths of equal cost and no unused path has a smaller cost – satisfies the variational inequality for any feasible flow z:

$$\langle c(z^{EQ}), z^{EQ} - z\rangle \leq 0 \ . \tag{3}$$

A simple proof of this is provided in [31].

Correa et. al. [28] use this result to develop a general tool for finding price of anarchy and bicriteria. To that end, they introduce the following parameters:

$$\beta(c, v) := \max_{z\in\mathbb{R}^{2n}_{\geq 0}} \frac{\langle c(v) - c(z), z\rangle}{\langle c(v), v\rangle} \ ,$$
$$\beta(\mathcal{C}) := \sup_{c\in\mathcal{C}, v\in\mathcal{X}} \beta(c, v) \ ,$$

where 0/0=0 by definition, $\mathcal{C}$ represents a class of cost functions, and $\mathcal{X}$ is the set of feasible routings.

In the following theorem, we adapt Correa et. al's Theorem 4.2 [28] to when the cost function is not monotone. In the nonmonotone case $\beta(\mathcal{C})$ can be greater than 1, leading to an unbounded price of anarchy. For completeness, we overview the proof of Theorem 4.2 in [28].

Theorem 1: Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}$ of nonseparable nonmonotone but elementwise monotone cost functions.

(a) If z^{OPT} is a social optimum for this game, and $\beta(\mathcal{C}) < 1$, then $C(z^{EQ}) \leq (1-\beta(\mathcal{C}))^{-1} C(z^{OPT})$.

(b) If w^{OPT} is a social optimum for the same game with $1+\beta(\mathcal{C})$ times as many players of each type, then $C(z^{EQ}) \leq C(w^{OPT})$.

Proof: To prove part (a),

$$\begin{aligned}\langle c(z^{EQ}), z\rangle &= \langle c(z), z\rangle + \langle c(z^{EQ}) - c(z), z\rangle \\ &\leq \langle c(z), z\rangle + \beta(c, z^{EQ})\langle c(z^{EQ}), z^{EQ}\rangle \\ &\leq C(z) + \beta(\mathcal{C}) C(z^{EQ}) \end{aligned} \tag{4}$$

and $C(z^{EQ}) \leq \langle c(z^{EQ}), z\rangle$. Completing the proof requires that $\beta(\mathcal{C}) \leq 1$.

To prove part (b), element-wise monotonicity implies the feasibility of $(1+\beta(\mathcal{C}))^{-1} w^{OPT}$, and using (3),

$$\langle c(z^{EQ}), z^{EQ}\rangle \leq \langle c(z^{EQ}), (1+\beta(\mathcal{C}))^{-1} w^{OPT}\rangle \ . \tag{5}$$

Then,

$$\begin{aligned} C(z^{EQ}) &= (1+\beta(\mathcal{C}))\langle c(z^{EQ}), z^{EQ}\rangle \\ &\quad - \beta(\mathcal{C})\langle c(z^{EQ}), z^{EQ}\rangle & (6)\\ &\leq (1+\beta(\mathcal{C}))\langle c(z^{EQ}), (1+\beta(\mathcal{C}))^{-1} w^{OPT}\rangle \\ &\quad - \beta(\mathcal{C})\langle c(z^{EQ}), z^{EQ}\rangle & (7)\\ &\leq C(w^{OPT}) \ , & (8)\end{aligned}$$

where (7) uses (5) and (8) uses (4). ■

B. Pairwise Separable Costs

We now present a bound for the price of anarchy and bicriteria for the pairwise separable affine cost function when k, the maximum degree of asymmetry of the cost function, is bounded. In particular, when $k < 4$, the price of anarchy is bounded, and the bicriteria is bounded for any k. This is formalized as follows:

Theorem 2: Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}_k$ of affine, pairwise separable, nonmonotone, elementwise monotone cost functions, where k parameterizes the maximum degree of asymmetry in the cost functions.

(a) If z^{OPT} is a social optimum for this game, and $k < 4$, then $C(z^{EQ}) \leq \frac{4}{4-k} C(z^{OPT})$.

(b) If w^{OPT} is a social optimum for the same game with $1+\frac{k}{4}$ times as many players of each type, then $C(z^{EQ}) \leq C(w^{OPT})$.

Proof: To prove this, we will show $\beta(\mathcal{C}_k) \leq \frac{k}{4}$ and then apply Theorem 1. For ease of notation, let $z^{EQ} \triangleq [x_1^* \quad y_1^* \quad x_2^* \quad y_2^* \quad \ldots \quad x_n^* \quad y_n^*]$.

Without loss of generality, and with a slight abuse of notation, we order the roads such that for $1 \leq i \leq \ell$, $c_i(x_i, y_i) = k_i a_i x_i + a_i y_i$ and for roads $\ell < i \leq n$, $c_i(x_i, y_i) = a_i x_i + k_i a_i y_i$, where $k_i \geq 1$. Then,

$$\begin{aligned}\beta(c, z^*) &= \max_{z\in\mathbb{R}^{2n}_{\geq 0}} \frac{\langle c(z^*) - c(z), z\rangle}{\langle c(z^*), z^*\rangle} \\ &\leq \frac{\max_{z\in\mathbb{R}^{2n}_{\geq 0}} \langle A(z^* - z), z\rangle}{\langle Az^*, z^*\rangle} \\ &= \frac{\sum_{i=1}^{\ell} a_i \max_{x_i, y_i \geq 0} (k_i(x_i^* - x_i) + (y_i^* - y_i))(x_i + y_i)}{\langle Az^*, z^*\rangle} \\ &\quad + \frac{\sum_{i=\ell+1}^{n} a_i \max_{x_i, y_i \geq 0} ((x_i^* - x_i) + k_i(y_i^* - y_i))(x_i + y_i)}{\langle Az^*, z^*\rangle} . \end{aligned} \tag{9}$$

We will bound the first term in (9), and the same can be done for the second term as well. Denote the inner term γ, so $\gamma(x_i, y_i) = (k_i(x_i^* - x_i) + (y_i^* - y_i))(x_i + y_i)$. This term is not concave, but is concave with respect to both x_i and y_i individually. Then, we use $f(x_i)$ to denote the function that maximizes γ with respect to y_i by solving $\frac{\partial\gamma}{\partial y_i}(x_i, y_i) = 0$, and $g(y_i)$ to denote the function that maximizes γ with respect to x_i by solving $\frac{\partial\gamma}{\partial x_i}(x_i, y_i) = 0$. This yields

$$f(x_i) = \frac{k_i x_i^* + y_i^*}{2} - \frac{k_i + 1}{2} x_i \ ,$$
$$g(y_i) = \frac{k_i x_i^* + y_i^*}{2k_i} - \frac{k_i + 1}{2k_i} y_i \ .$$

Then, for any fixed x_i, the optimal y_i is determined by $y_i = f(x_i)$, and for any fixed y_i, the optimal x_i is determined by $x_i = g(y_i)$. Then, define $\tilde{x_i}$ and $\tilde{y_i}$ as follows:

$$\tilde{x_i} = \operatorname*{argmax}_{x_i\geq 0,\ f(x_i)\geq 0} \gamma(x_i, f(x_i)) \ ,$$
$$\tilde{y_i} = \operatorname*{argmax}_{y_i\geq 0,\ g(y_i)\geq 0} \gamma(g(y_i), y_i) \ .$$

We see that $\gamma(x_i, f(x_i))$ and $\gamma(g(y_i), y_i)$ are convex, and $\gamma(\tilde{x}_i, f(\tilde{x}_i)) \geq \gamma(g(\tilde{y}_i), \tilde{y}_i)$, where $\tilde{x}_i = 0$. Therefore,

$$\max_{x_i \geq 0, y_i \geq 0} \gamma(x_i, y_i) = \gamma(\tilde{x}_i, f(\tilde{x}_i)) = \frac{(k_i x_i^* + y_i^*)^2}{4} \ .$$

After applying a similar analysis for roads $\ell < i \leq n$,

$$\begin{aligned} &\beta(c, z^*) \\ &\leq \frac{1}{4} \frac{\sum_{i=1}^{\ell} \rho_i (k_i x_i^* + y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (x_i^* + k_i y_i^*)}{\sum_{i=1}^{\ell} \rho_i (x_i^* + y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (x_i^* + y_i^*)} \\ &= \frac{k}{4} \frac{\sum_{i=1}^{\ell} \rho_i (k_i x_i^* + y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (x_i^* + k_i y_i^*)}{\sum_{i=1}^{\ell} \rho_i (k x_i^* + k y_i^*) + \sum_{i=\ell+1}^{n} \sigma_i (k x_i^* + k y_i^*)} \\ &\leq \frac{k}{4} \ , \end{aligned} \tag{10}$$

where $\rho_i \triangleq a_i(k_i x_i^* + y_i^*)$ and $\sigma_i \triangleq a_i(x_i^* + k_i y_i^*)$. The fact that $\frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n v_i} \leq 1$ when $0 \leq w_i \leq v_i$ implies Equation (10), since $k \geq k_i \geq 1 \ \forall i$. We apply Theorem 1 to find a price of anarchy bound of $\frac{4}{4-k}$ and bicriteria bound of $1 + \frac{k}{4}$. ■

C. Nonseparable costs

Having discussed pairwise separable costs (Definition 2), where the delay on each road depends only on the vehicles on that road, we now consider nonseparable costs (Definition 3). As an example, consider a series of roads, each one feeding into the next; if one road is fully congested, this will increase the delay on the roads feeding it, resulting in cascading congestion. Another scenario of nonseparable costs is when intersecting streets affect the traffic on each other [23], such as in a signalized intersection that senses traffic and adjusts its duty cycle accordingly. In that case, the volume of traffic on a road will affect the delay on the perpendicular road.

To put this in more concrete terms, consider a road feeding into another narrower road. We model the congestion on the second road as comparatively affecting that on the first road by a factor of μ. This results in a cost function of

$$c(z) = \begin{bmatrix} k_1 a_1 & a_1 & \mu k_2 a_2 & \mu a_2 \\ k_1 a_1 & a_1 & \mu k_2 a_2 & \mu a_2 \\ 0 & 0 & k_2 a_2 & a_2 \\ 0 & 0 & k_2 a_2 & a_2 \end{bmatrix} z + b \ .$$

With this motivation, we consider the affine cost functions $c(x) = Ax + b$, where A is no longer a 2x2 block-diagonal matrix. We consider the case that A can be written as the sum of Q, a (2x2) block diagonal matrix with strictly positive block diagonal entries, and P, a positive definite matrix.[2]

[2]Note that if P is a diagonal dominant mapping, *i.e.* $P_{ii} > \frac{1}{2}\sum_{j \neq i} |P_{ij} + P_{ji}|$, then it is positive definite [31]. In this case, in order to also guarantee that the block diagonal components of Q have strictly positive entries, we require

$$A_{ii} > \frac{1}{2} \sum_{j \neq i, i-1} |A_{ij} + A_{ji}| \text{ for } i \text{ even} \ ,$$

$$A_{ii} > \frac{1}{2} \sum_{j \neq i, i+1} |A_{ij} + A_{ji}| \text{ for } i \text{ odd} \ .$$

This, however, is a sufficient but not necessary condition.

We describe the bounds we can establish under these conditions in the following theorem:

Theorem 3: Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost function $c(z) = Az + b$. Suppose A can be split into Q, which is a (2x2) block diagonal matrix with strictly positive entries on the block diagonal, and P, which is positive definite, such that $A = Q + P$. Let k be the maximum degree of asymmetry for the cost function defined by Q.

(a) If z^{OPT} is a social optimum for this game, and if $k < 4$, then $C(z^{EQ}) \leq (\frac{4}{4-k} + \eta^2) C(z^{OPT})$, where $\eta^2 = \lambda_{max}(S^{-1/2} P^T S^{-1} P S^{-1/2})$ and $S = (P + P^T)/2$.

(b) If w^{OPT} is a social optimum for the same game with $2 + \frac{k}{4}$ times as many players of each type, then $C(z^{EQ}) \leq C(w^{OPT})$.

Proof: For part (a), we split the price of anarchy into two components, as

$$\begin{aligned} \frac{C^{EQ}}{C^{OPT}} &= \sup_{z \in \mathcal{X}} \frac{(Az^{EQ} + b)^T z^{EQ}}{(Az + b)^T z} \\ &= \sup_{z \in \mathcal{X}} \frac{((Q+P)z^{EQ} + b)^T z^{EQ}}{((Q+P)z + b)^T z} \\ &\leq \sup_{z \in \mathcal{X}} \frac{(Qz^{EQ} + b)^T z^{EQ}}{(Qz + b)^T z} + \sup_{z \in \mathcal{X}} \frac{(Pz^{EQ})^T z^{EQ}}{(Pz)^T z} \end{aligned} \tag{11}$$

$$\leq \frac{1}{1 - \beta(\mathcal{C}_k)} + \eta^2 \tag{12}$$

$$= \frac{4}{4-k} + \eta^2 \ . \tag{13}$$

Inequality (11) follows from all latencies being nonnegative, (12) follows from [28] and [27] (see the comment on page 2 about the price of anarchy for costs with no constant term), and the (13) is proved in the proof of Theorem 2.

For part (b), we use the same notion of $\beta(\mathcal{C})$ as in the proofs for Theorems 1 and 2, as follows:

$$\begin{aligned} \beta(c, v) &= \max_{z \in \mathbb{R}^{2n}_{\geq 0}} \frac{\langle c(v) - c(z), z \rangle}{\langle c(v), v \rangle} \\ &= \frac{\max_{z \in \mathbb{R}^{2n}_{\geq 0}} \langle (Q+P)(v-z), z \rangle}{\langle (Q+P)v + b, v \rangle} \\ &\leq \frac{\max_{z \in \mathbb{R}^{2n}_{\geq 0}} \langle Q(v-z), z \rangle}{\langle (Q+P)v + b, v \rangle} + \frac{\max_{z \in \mathbb{R}^{2n}_{\geq 0}} \langle P(v-z), z \rangle}{\langle (Q+P)v + b, v \rangle} \\ &\leq \frac{\max_{x \in \mathbb{R}^{2n}_{\geq 0}} \langle Q(v-z), z \rangle}{\langle Qv + b, v \rangle} + \frac{\max_{z \in \mathbb{R}^{2n}_{\geq 0}} \langle P(v-z), z \rangle}{\langle Pv + b, v \rangle} \\ &= \beta(c_1, v) + \beta(c_2, v) \end{aligned}$$

Here c_1 and c_2 represent cost functions drawn from $\mathcal{C}_k$ and $\tilde{\mathcal{C}}$, respectively, where k is the maximum degree of asymmetry of the cost function $c(z) = Qz + b$ and $\tilde{\mathcal{C}}$ denotes the set of monotone cost functions.

De Palma and Nesterov [31] show that a cost function $c(z)$ is monotone if $c'(z)$ is positive definite. Furthermore, Correa *et. al.* show that a class $\mathcal{C}$ consisting of monotone cost functions has $\beta(\mathcal{C}) \leq 1$. This is easily demonstrated as

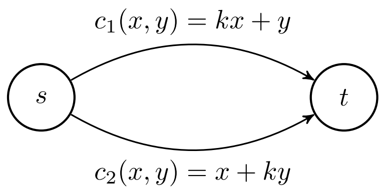

Fig. 2: Example of a road network with two-sided asymmetry.

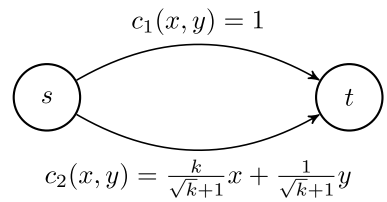

Fig. 3: Example of a road network with one-sided asymmetry.

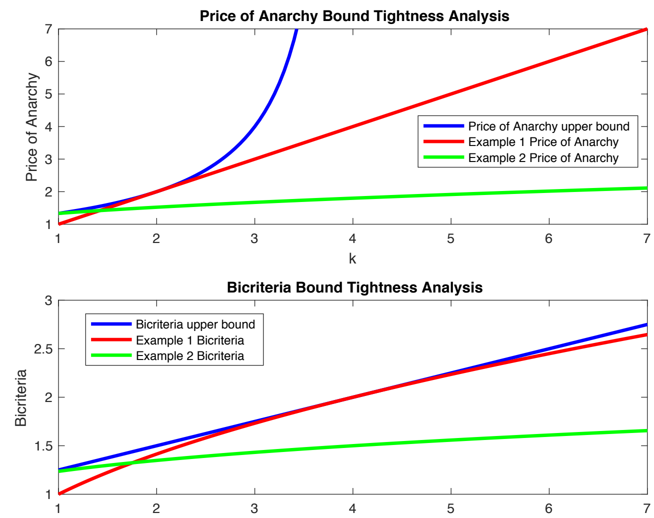

Fig. 4: Tightness of the bounds for price of anarchy and bicriteria. The PoA bound is tight for $k = 1, 2$ and the bicriteria bound is tight for $k = 4$.

follows. Using (1) with $z, v \in \mathbb{R}^{2n}_{\geq 0}$,

$$1 \geq \frac{\langle c(v) - c(z), z\rangle}{\langle c(v) - c(z), v\rangle} \geq \frac{\langle c(v) - c(z), z\rangle}{\langle c(v), v\rangle} \geq \beta(c, v) \; .$$

Because of this,

$$\begin{aligned}\beta(\mathcal{C}) &= \sup_{c\in\mathcal{C}, v\in\mathcal{X}} \beta(c, v) \\ &\leq \sup_{c\in\mathcal{C}_k, v\in\mathcal{X}} \beta(c, v) + \sup_{c\in\tilde{\mathcal{C}}, v\in\mathcal{X}} \beta(c, v) \\ &\leq \frac{k}{4} + 1 \; .\end{aligned}$$

Here $\tilde{\mathcal{C}}$ denotes monotone cost functions. Applying Theorem 1 completes the proof. ■

V. TIGHTNESS OF THE BOUND

In this section, we discuss the tightness of the bound derived in Section IV-B. In Section V-A we provide two examples: Example 2 shows that our price of anarchy is tight for $k = 2$ and our bicriteria bound is tight for $k = 4$ when there can be two-sided asymmetry, *i.e.* k_i can be greater or less than 1. In a more realistic scenario, we expect autonomous vehicles to result in the same amount or less congestion than regular cars for all roads. In light of this, we provide Example 2 of *one-sided* asymmetry, in which $k_i \geq 1 \; \forall i$. In Section V-B, we discuss the tightness of the bound with respect to both of these scenarios.

A. Examples

Example 2: Consider the traffic network in Fig. 2, which is parameterized by the degree of asymmetry, k. We wish to transport 1 unit regular traffic and 1 unit smart traffic across the network.

The worst-case Nash equilibrium has all regular traffic on the top link and all the smart traffic on the bottom link, for a cost of $C^{EQ} = 2k$. The optimal routing has this routing reversed, for a cost of $C^{OPT} = 2$. This gives us $\frac{C^{EQ}}{C^{OPT}} = k$.

We find the bicriteria by finding how much traffic we could optimally route for a cost of $2k$. Consider p units regular and p units of smart vehicles, which would have optimal routing cost $2p^2$. Setting $2p^2 = 2k$, we find the bicriteria is $\sqrt{k}$.

Example 3: Consider the traffic network in Fig. 3, which is parameterized by k. Here we wish to transport $\frac{1}{\sqrt{k}}$ units regular traffic and 1 unit smart traffic across the network.

At the Wardrop Equilibrium, all traffic will take the bottom route for a delay of 1, which gives us cost $C^{EQ} = \frac{1}{\sqrt{k}} + 1$. In optimal routing we have regular traffic on top and smart traffic on the bottom. This gives us $C^{OPT} = \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k}+1}$, giving us a PoA of $1 + \frac{k}{2\sqrt{k}+1}$.

We find the bicriteria by setting the cost of routing p times as much traffic optimally equal to the original cost at equilibrium. This gives us $p = \frac{(-1+\sqrt{1+4\sqrt{k}})(1+\sqrt{k})}{2\sqrt{k}}$.

B. Discussion

We begin by discussing the price of anarchy. Our bound for price of anarchy is $\frac{4}{4-k}$, and example 2 shows a price of anarchy of k and example 3 shows a price of anarchy of $1 + \frac{k}{2\sqrt{k}+1}$. For the bicriteria, our bound is $1 + \frac{k}{4}$. Example 1 provides a bicriteria of $\sqrt{k}$ and example 2 has a bicriteria that scales with $k^{1/4}$.

When $k = 1$, price of anarchy bound recovers the classical bound found in [21]. Further, the examples show that the price of anarchy bound is tight for $k = 2$ and the bicriteria bound is tight for $k = 4$.

Figure 4 illustrates these comparisons. In both cases, our upper bound diverges from these lower bounding examples for large k. Therefore, it is unknown if our bound is tight in that regime. However, realistic circumstances lead to $k \approx 2.5$, which is in the near-tight region for both price of anarchy and bicriteria.

It is worth noting that under the construction in [28] and in Theorem 1, there can be no bound on the price of anarchy for networks with $k \geq 4$. Observe that in Example 2 for $k = 4$, the bicriteria is 2. This means that $\beta(\mathcal{C}_{k=4}) \geq 1$, so the bound on the price of anarchy does not hold.

VI. CONCLUSIONS

In this paper, we have presented pairwise separable and nonseparable cost functions for traffic networks under mixed autonomy. We demonstrate that the price of anarchy and bicriteria is unbounded without constraints on the asymmetry in the difference in how the additon of smart and regular vehicles affects congestion. We then established bounds for the price of anarchy and bicriteria, parameterized by the degree of asymmetry of the network, for both the case of pairwise separable and nonseparable costs, under certain conditions. We analyze the tightness of the bounds for the pairwise separable case and demonstrate that they are tight for certain degrees of asymmetry of the network.

REFERENCES

[1] *Beyond Traffic 2045: Trends and Choices*. US Department of Transportation, 2015.

[2] J. Lioris, R. Pedarsani, F. Y. Tascikaraoglu, and P. Varaiya, "Platoons of connected vehicles can double throughput in urban roads," *Transportation Research Part C: Emerging Technologies*, vol. 77, pp. 292–305, 2017.

[3] S. E. Shladover, "Longitudinal control of automated guideway transit vehicles within platoons," *Journal of Dynamic Systems, Measurement, and Control*, vol. 100, no. 4, pp. 302–310, 1978.

[4] S. Darbha and K. Rajagopal, "Intelligent cruise control systems and traffic flow stability," *Transportation Research Part C: Emerging Technologies*, vol. 7, no. 6, pp. 329–352, 1999.

[5] J. Yi and R. Horowitz, "Macroscopic traffic flow propagation stability for adaptive cruise controlled vehicles," *Transportation Research Part C: Emerging Technologies*, vol. 14, no. 2, pp. 81–95, 2006.

[6] R. Pueboobpaphan and B. van Arem, "Driver and vehicle characteristics and platoon and traffic flow stability: Understanding the relationship for design and assessment of cooperative adaptive cruise control," *Transportation Research Record: Journal of the Transportation Research Board*, no. 2189, pp. 89–97, 2010.

[7] G. Orosz, "Connected cruise control: modelling, delay effects, and nonlinear behaviour," *Vehicle System Dynamics*, pp. 1–30, 2016.

[8] L. Jin, M. Čičić, S. Amin, and K. H. Johansson, "Modeling impact of vehicle platooning on highway congestion: A fluid queuing approach,"

[9] A. Askari, D. A. Farias, A. A. Kurzhanskiy, and P. Varaiya, "Measuring impact of adaptive and cooperative adaptive cruise control on throughput of signalized intersections," *arXiv preprint arXiv:1611.08973*, 2016.

[10] J. Vander Werf, S. Shladover, M. Miller, and N. Kourjanskaia, "Effects of adaptive cruise control systems on highway traffic flow capacity," *Transportation Research Record: Journal of the Transportation Research Board*, no. 1800, pp. 78–84, 2002.

[11] B. Van Arem, C. J. Van Driel, and R. Visser, "The impact of cooperative adaptive cruise control on traffic-flow characteristics," *IEEE Transactions on Intelligent Transportation Systems*, vol. 7, no. 4, pp. 429–436, 2006.

[12] R. Jiang, M.-B. Hu, B. Jia, R. Wang, and Q.-S. Wu, "Phase transition in a mixture of adaptive cruise control vehicles and manual vehicles," *The European Physical Journal B*, vol. 58, no. 2, pp. 197–206, 2007.

[13] Y.-M. Yuan, R. Jiang, M.-B. Hu, Q.-S. Wu, and R. Wang, "Traffic flow characteristics in a mixed traffic system consisting of ACC vehicles and manual vehicles: A hybrid modelling approach," *Physica A: Statistical Mechanics and its Applications*, vol. 388, no. 12, pp. 2483–2491, 2009.

[14] A. Kesting, M. Treiber, and D. Helbing, "Enhanced intelligent driver model to access the impact of driving strategies on traffic capacity," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 368, no. 1928, pp. 4585–4605, 2010.

[15] G. M. Arnaout and S. Bowling, "A progressive deployment strategy for cooperative adaptive cruise control to improve traffic dynamics," *International Journal of Automation and Computing*, vol. 11, no. 1, pp. 10–18, 2014.

[16] D. Lazar, S. Coogan, and R. Pedarsani, "Capacity modeling and routing for traffic networks with mixed autonomy," in *IEEE 56th Annual Conference on Decision and Control (CDC)*, IEEE, 2017.

[17] J. Wardrop, "Some theoretical aspects of road traffic research," in *Inst Civil Engineers Proc London/UK/*, 1900.

[18] S. C. Dafermos and F. T. Sparrow, "The traffic assignment problem for a general network," *Journal of Research of the National Bureau of Standards B*, vol. 73, no. 2, pp. 91–118, 1969.

[19] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," in *Annual Symposium on Theoretical Aspects of Computer Science*, pp. 404–413, Springer, 1999.

[20] C. Papadimitriou, "Algorithms, games, and the internet," in *Proceedings of the thirty-third annual ACM symposium on Theory of computing*, pp. 749–753, ACM, 2001.

[21] T. Roughgarden and É. Tardos, "How bad is selfish routing?," *Journal of the ACM (JACM)*, vol. 49, no. 2, pp. 236–259, 2002.

[22] T. Roughgarden, "Selfish routing," tech. rep., Cornell Univ. Ithaca, NY Dept. of Computer Science, 2002.

[23] J. R. Correa and N. E. Stier-Moses, "Wardrop equilibria," *Wiley encyclopedia of operations research and management science*, 2011.

[24] T. Roughgarden, *Selfish routing and the price of anarchy*, vol. 174. MIT press Cambridge, 2005.

[25] T. Roughgarden, "The price of anarchy is independent of the network topology," *Journal of Computer and System Sciences*, vol. 67, no. 2, pp. 341–364, 2003.

[26] C. K. Chau and K. M. Sim, "The price of anarchy for non-atomic congestion games with symmetric cost maps and elastic demands," *Operations Research Letters*, vol. 31, no. 5, pp. 327–334, 2003.

[27] G. Perakis, "The price of anarchy under nonlinear and asymmetric costs," *Mathematics of Operations Research*, vol. 32, no. 3, pp. 614–628, 2007.

[28] J. R. Correa, A. S. Schulz, and N. E. Stier-Moses, "A geometric approach to the price of anarchy in nonatomic congestion games," *Games and Economic Behavior*, vol. 64, no. 2, pp. 457–469, 2008.

[29] S. Sekar, L. Zheng, L. Ratliff, and B. Zhang, "Uncertainty in multicommodity routing networks: When does it help?," *arXiv preprint arXiv:1709.08441*, 2017.

[30] A. C. Pigou, "The economics of welfare," *McMillan&Co., London*, 1920.

[31] A. De Palma and Y. Nesterov, "Optimization formulations and static equilibrium in congested transportation networks," tech. rep., 1998.

[32] A. Askari, D. A. Farias, A. A. Kurzhanskiy, and P. Varaiya, "Effect of adaptive and cooperative adaptive cruise control on throughput of signalized arterials," in *Intelligent Vehicles Symposium (IV), 2017 IEEE*, pp. 1287–1292, IEEE, 2017.

[33] V. Milanés, S. E. Shladover, J. Spring, C. Nowakowski, H. Kawazoe, and M. Nakamura, "Cooperative adaptive cruise control in real traffic situations," *IEEE Transactions on Intelligent Transportation Systems*, vol. 15, no. 1, pp. 296–305, 2014.

[34] M. J. Smith, "The existence, uniqueness and stability of traffic equilibria," *Transportation Research Part B: Methodological*, vol. 13, no. 4, pp. 295–304, 1979.