Strong Integral Input-to-State Stability in Dynamical Flow Networks

Gustav Nilsson and Samuel Coogan

***Abstract*— Dynamical flow networks are vital in modeling many networks, such as transportation networks, distribution networks, and queuing networks. While the flow dynamics in such networks follow the conservation of mass on the links, the outflow from each link is often non-linear due to the actual flow dynamics, flow capacity constraints, and simultaneous service constraints. Such non-linear constraints imply a limit on the magnitude of exogenous inflows that a dynamical flow network can handle. This paper shows how the Strong integral Input-to-State Stability (Strong iISS) property allows for quantifying the effects of the exogenous inflow on the flow dynamics. The Strong iISS property enables a unified stability analysis of classes of dynamical flow networks that were only partly analyzable before, such as multi-commodity flow networks, networks with cycles, and networks with non-monotone flow dynamics. We first present sufficient conditions on the maximum magnitude of exogenous inflow to guarantee input-to-state stability for a dynamical flow network. We next exemplify the conditions by applying them to existing dynamical flow network models, specifically, fluid queuing models and multi-commodity flow models.**

I. Introduction

Dynamical network flow models have recently become popular to model physical network flows such as transportation networks [1], [2], [3] as well as non-physical processing networks such as queuing systems [4]. One frequent common component for those networks is a limitation on the magnitude of exogenous inflow the networks can handle. The dynamics of such networks is often non-linear, both due to the physical flow dynamics itself and saturation in the service rates. Thus, classical techniques such as linear system analysis are not enough to analyze these systems' stability properties.

In many of the aforementioned applications, the goal is to keep link densities or queues bounded. Usually, this is possible as long as the exogenous inflows stay below a certain threshold. The Strong Integral Input-to-State property (Strong iISS), was introduced in [5] to combine integral input-to-state stability with input-to-state stability for small inputs, and to determine when the input is small enough to guarantee the latter. Although those two properties align naturally with the expected behavior of dynamical flow networks, to the best of the author's knowledge, the results in [5] have not been exploited for studying dynamical flow networks in a general setting with features such as cycles, multi-commodity flows, and non-monotone flow dynamics. Apart from having the desired property of guaranteeing stability when the exogenous inflow to the network is lower than a certain threshold, the Strong iISS property also imposes that if the exogenous inflow becomes zero at a certain time, the total mass in the network will also eventually converge to zero. For many applications, this is an essential property. For example, in transportation networks, a correct traffic signal control solution should allow for the vehicles to eventually leave the network.

Previously, the stability properties of dynamical flow networks have been analyzed by utilizing monotonicity properties of the dynamics to construct a contraction argument [6], [7], extending the state-space and using a mixed-monotonicity argument together with a uniqueness of equilibrium argument [8], a contraction argument based on the Jacobian of the system dynamics [9], utilizing passivity theory [10] or constructing specific entropy-like Lyapunov functions [11].

While the properties of the flow networks are well understood when the dynamics is monotone, and in some particular cases mixed-monotone, it is not trivial how to analyze the stability of dynamical flow networks that do not have the monotonicity property. As noticed in [12], when extending the flow network models to multi-commodity flows, the monotonicity property is usually lost. Another example when the system's monotonicity property is lost is when a feedback controller can serve more than one queue simultaneously, and the service is split in proportion to the demand in all queues that are served simultaneously [13]. This situation is common in many applications, such as when controlling traffic signals in a transportation network.

The stability analysis in this paper partly relies on a special variant of sum-separable Lyapunov functions. Such Lyapunov functions have previously been combined with monotonicity properties, e.g., [9], [14]. The Lyapunov function is also be based on a transformation involving the inverse of the routing matrix for the network. This transformation has previously been utilized to obtain monotonicity properties for tree-like flow networks in [15].

The rest of the paper is organized as follows: The remainder of this section is devoted to introducing some basic notation that will be used throughout the paper. In Section II, we present the dynamical flow network model, together with a few general model assumptions. We also show that under those mild assumptions, dynamical flow networks are always integral input-to-state stable. In Section III we present sufficient conditions on the exogenous inflow for the dynamical flow networks to be input-to-state stable, along

The authors are with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, 30332, USA. {gustav.nilsson, sam.coogan}@gatech.edu. S. Coogan is also with the School of Civil and Environmental Engineering, Georgia Institute of Technology. This work was supported by the National Science Foundation under grant #1749357 and the Air Force Office of Scientific Research under award FA9550-19-1-0015.

with conditions when the bound is tight for a local network. In Section IV, we illustrate how the stability theory can be applied to existing models for dynamical flow networks, namely multi commodity flow and dynamical networks with time-varying exogenous inflows. The paper concludes with some ideas for future research.

A. Notation

We let $\mathbb{R}_+$ denote the non negative reals. For a finite set $\mathcal{A}$, $\mathbb{R}_+^{\mathcal{A}}$ denote the set of non-negative vectors indexed by $\mathcal{A}$. For vectors $w, x \in \mathbb{R}_+^n$ such that $w > 0$, we introduce the weighted ℓ_1 norm as $\|x\|_w = w^T x$. The all-one vector is denoted by $\mathbf{1}$. A function $\mu(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ if it is strictly increasing, $\mu(0) = 0$, and $\lim_{x \to +\infty} \mu(x) = +\infty$. A function $\beta(x,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said be of class $\mathcal{KL}$ if $\beta(0,t) = 0$ for all t, it is strictly increasing in x for each fixed t, and it is decreasing in t for each fixed x and $\lim_{t \to +\infty} \beta(x,t) \to 0$.

II. Model

We model a dynamical flow network as a capacited directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, c)$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ the set of links or edges, and $c \in \mathbb{R}_+^{\mathcal{E}}$ such that $c_i > 0$ for all $i \in \mathcal{E}$ is the maximum outflow capacity of each link. For simplicity of notation, we will denote the matrix $C = \mathrm{diag}(c)$ and the vector $\bar{c} = (1/c_i)_{i \in \mathcal{E}}$. For an link $e = (i,j) \in \mathcal{E}$, we let $\tau(e)$ denote the tail of the link, i.e., $\tau(e) = i$, and $\sigma(e)$ the head of the link, i.e., $\sigma(e) = i$. Moreover, we let $\mathcal{E}_v$ denote the subset of incoming links to node $v \in \mathcal{V}$, formally $\mathcal{E}_v = \{i \in \mathcal{E} \mid \sigma(i) = v\} \subset \mathcal{E}$.

In the flow network, mass flows along the links $\mathcal{E}$. Therefore, the network's state $x \in \mathcal{X} \subset \mathbb{R}_+^{\mathcal{E}}$ is the vector of masses on all the links in the network, and $\mathcal{X}$ denotes the state space. To a subset of the links, there is possibly time varying exogenous inflows, which we denote by $\lambda(t) \in \mathbb{R}_+^{\mathcal{E}}$.

To model how mass propagates in the network, we introduce the routing matrix $R \in \mathbb{R}_+^{\mathcal{E} \times \mathcal{E}}$, where each element $0 \le R_{ij} \le 1$ is the fraction of outflow from link i that proceeds to link j. By topological constraints from the graph $\mathcal{G}$, $R_{ij} = 0$ whenever $\sigma(i) \neq \tau(j)$. The routing matrix is substochastic, i.e, for each $j \in \mathcal{E}$, $\sum_i R_{ji} \le 1$ where the quantity $1 - \sum_i R_{ji}$ is the fraction of mass that leaves the network after flowing out from link j. Throughout the paper, we make the following assumption on the routing matrix, stating that there is a path from every link to a link where the flow can leave the network:

Assumption 1: The routing matrix R is outflow connected, i.e., for every link $i \in \mathcal{E}$, there exist a link $j \in \mathcal{E}$ such that $\sum_\ell R_{j\ell} < 1$ and a path $(e_1, \dots, e_l)$ with $e_1 = i$ and $e_l = j$ such that $\Pi_{1 \le h < l} R_{e_h, e_{h+1}} > 0$.

Assumption 1 implies that the spectral radius of R is less than one [16], and hence the matrix $(I - R^T)$ is invertible. Moreover, the inverse is computed as

$$(I - R^T)^{-1} = \sum_{k \ge 0} (R^T)^k = I + R^T + (R^T)^2 + \dots . \quad (1)$$

The outflow from each link in the network is limited by a flow function that determines the fraction of the link's flow capacity c_i that is available at each moment. We let $f_i(x) : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}_+$ denote this state-dependent outflow fraction for each link. The outflow depends on the state (i.e., mass) of the link itself, but, depending on the application, it can also depend on the state of neighboring links. We will throughout the paper, assume the following properties of the flow functions.

Assumption 2: The flow functions $f_i(x)$ for all $i \in \mathcal{E}$ are continuous and such that:

i) $f_i(x) \le 1$ for all $x \in \mathcal{X}$ and all $i \in \mathcal{E}$ and
ii) $f_i(x) = 0$ if and only if $x_i = 0$.

The last part of the assumption ensures that the outflow is only zero when there is zero mass present on the link. This property ensures set invariance with respect to the state space $\mathcal{X}$, and the "only if" direction further ensures that the flow function is work-conservative. In other words, there will not be zero outflow if there is mass present on the link. As we observe later, this property is also needed to ensure that all the flow will eventually leave the network.

Remark 1: In difference to the related work in e.g., [7], we do not assume any monotonicity properties of the flow functions $f_i(x)$. □

The dynamics of the flow network follows the conservation of mass: the change of mass on each link is equal to sum of the exogenous inflow and upstream outflows minus the outflow from the link itself. That is,

$$\dot{x}_i = \lambda_i(t) + \sum_{j \in \mathcal{E}} R_{ji} c_j f_j(x) - c_i f_i(x), \quad \forall i \in \mathcal{E} .$$

The dynamics are also expressed in vector form as

$$\dot{x} = \lambda(t) - (I - R^T) C f(x) \quad (2)$$

where $f(x)$ denotes the vector consisting of all the flow functions for the links, i.e., $f(x) = (f_i(x))_{i \in \mathcal{E}}$.

We say that the system (2) is stable when the following two conditions hold, which is a direct consequence of input-to-state stability [17]:

Definition 1 (Stability): A dynamical flow network with exogenous inflow $\lambda(t)$ is stable if there exists a constant $D > 0$ such that for all $t \ge 0$, $\|x(t)\| < D$. Moreover, if there exists $t' > 0$ such that $\lambda(t) = 0$ for all $t \ge t'$, then $\lim_{t \to +\infty} x(t) = 0$ for the network to be stable.

While the first part of the definition ensures that the state remains bounded, the second part ensures that when the exogenous inflow is zero, all the mass in the network will eventually leave.

We begin by establishing a fundamental but conservative bound on the state of a dynamical flow network with exogenous inflow. This bound ensures that the total amount of mass in the dynamical flow network will always be bounded by its initial state and the amount of exogenous inflow to the network and does not require Assumption 2.

Proposition 1: For a dynamical flow network (2), that satisfies Assumption 1, let $a(t) = (I - R^T)^{-1}\lambda(t)$. Then,

$$x_i(t) \le \int_0^t a_i(s)ds + \xi_i\,, \quad \forall i \in \mathcal{E}\,,$$

where $\xi = (I - R^T)^{-1}x(0)$.

Proof Let $\hat{x} = (I - R^T)^{-1}x$. Then

$$\dot{\hat{x}} = (I - R^T)^{-1}\lambda(t) - f(x) = a(t) - f(x)$$

and $\hat{x}(0) = (I - R^T)^{-1}x(0)$. Since $f(x) \ge 0$ it holds that $\dot{\hat{x}}_i \le a_i(t)$ for all $i \in \mathcal{E}$, and hence

$$\hat{x}_i(t) \le \int_0^t a_i(s)ds + \hat{x}_i(0)\,, \quad \forall i \in \mathcal{E}\,. \tag{3}$$

Observe that $\hat{x}(t) \ge 0$ for all $t \ge 0$. This since $(I - R^T)^{-1} = \sum_{k \ge 0}(R^T)^k$ will have all elements non-negative and both $\lambda \ge 0$ and $x(0) \ge 0$.

By transforming back to x, i.e., $x = (I - R^T)\hat{x}$, it then for each $i \in \mathcal{E}$ holds that

$$x_i(t) = \hat{x}_i(t) - \sum_j R_{ji}\hat{x}_j(t) \le \hat{x}_i(t) \le \int_0^t a_i(s)ds + \hat{x}_i(0)\,.$$

■

Remark 2: In (3), the term $\int_0^t a_i(s)ds$ indicates how much mass can possibly reach link $i \in \mathcal{E}$ from outside the network, and the term $\tilde{x}_i$ indicates how much mass can reach link i from inside the network. □

The bound in Proposition 1 is very general, since it accommodates cases when $f_i(x) = 0$ for $x_i > 0$. By assuming that Assumption 2 holds and exploiting the results for Strong iISS in [5], we can establish an alternative bound that is stronger since the dependence of the initial state will vanish with time.

Proposition 2: For a dynamical flow network (2) that satisfy Assumption 1 and 2 there exist $\beta \in \mathcal{KL}$ and $\mu_1, \mu_2 \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \left\|(I - R^T)^{-1}x(t)\right\|_{\bar{c}} &\le \beta(\left\|(I - R^T)^{-1}x(0)\right\|_{\bar{c}}, t) \\ &+ \mu_1\left(\int_0^t \mu_2(\left\|(I - R^T)^{-1}\lambda(s)\right\|_{\bar{c}})ds\right)\,, \end{aligned}$$

for all $t \ge 0$.

Proof Introduce the Lyapunov candidate

$$V(x) = \mathbf{1}^T C^{-1}(I - R^T)^{-1}x\,.$$

Clearly $V(x) \ge 0$ since R is assumed to be outflow connected in Assumption 1, and combined with the expression (1) it can be seen that all elements of R are non-negative and strictly positive on the diagonal. Moreover, for the same reason, $V(x) = 0$ if and only if $x = 0$. Hence $V(x)$ is a proper storage function and its drift is given by

$$\begin{aligned} \frac{dV}{dt} = \frac{\partial V}{\partial x}f(x) &= \mathbf{1}^T C^{-1}(I - R^T)^{-1}(\lambda - (I - R^T)Cf(x)) \\ &= \sum_{i \in \mathcal{E}} \frac{a_i(t)}{c_i} - \sum_{i \in \mathcal{E}} f_i(x)\,. \end{aligned}$$

Let $\gamma(x) = x$ and $W(x) = \sum_i f_i(x)$. Clearly, $\gamma \in \mathcal{K}_\infty$ and

$$\gamma(\left\|(I - R^T)^{-1}\lambda(t)\right\|_{\bar{c}}) = \sum_{i \in \mathcal{E}} \frac{a_i(t)}{c_i}\,.$$

Hence

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}f(x) = -W(x) + \gamma(\left\|(I - R^T)^{-1}x\right\|_{\bar{c}})\,.$$

Moreover, from Assumption 2 it follows that $W(x)$ is positive definite. Now, applying [5, Theorem 1], gives the bound. ■

Note that the previous proposition does not provide any guarantees that the dynamical flow network will be stable according to Definition 1. In the next section, we establish sufficient conditions on the exogenous inflow to ensure stability.

III. Stability Analysis

We start this section by establish a sufficient condition on the exogenous inflows for the dynamical flow network to be stable. We then prove that the sufficient condition is also necessary under an additional requirement on the flow functions in the special case of a local network.

A. Sufficient Condition for Stability

We begin with this paper's main result, where we establish a sufficient condition for stability of dynamical flow networks by using the strong iISS [5] theory.

Theorem 1: For a dynamical flow network (2) that satisfy Assumption 1 and 2, let $a(t) = (I - R^T)^{-1}\lambda(t)$. If

$$\operatorname*{ess\,sup}_{t \ge 0} \sum_i \frac{a_i(t)}{c_i} < \liminf_{\|x\| \to +\infty} \sum_i f_i(x)\,, \tag{4}$$

then there exists functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that the solution to the dynamical flow (2) satisfies

$$\begin{aligned} \left\|(I - R^T)^{-1}x(t)\right\|_{\bar{c}} &\le \beta(\left\|(I - R^T)^{-1}x(0)\right\|_{\bar{c}}, t) + \\ &\mu\left(\operatorname*{ess\,sup}_{t \ge 0} \left\|(I - R^T)^{-1}\lambda(t)\right\|_{\bar{c}}\right)\,. \end{aligned}$$

In particular, the dynamical flow network is stable.

Proof Using the same Lyapunov function as in Proposition 2, we observe that condition (4) is equivalent to

$$\operatorname*{ess\,sup}_{t \ge 0} \left\|(I - R^T)^{-1}\lambda(t)\right\|_{\bar{c}} < Q\,,$$

where $Q = \gamma^{-1}(W_\infty)$ with $W_\infty = \liminf_{\|x\| \to +\infty} W(x)$. The bound then follows from [5, Theorem 1]. ■

Remark 3: While this paper focuses on bounded flow functions f_i, it is possible to extend the theory to show that for unbounded flow functions dynamical flow networks are always stable. In particular, if we let $c_i = 1$, but violate part (ii) of Assumption 2 by instead requiring the flow functions to be unbounded such that

$$\liminf_{\|x\| \to +\infty} f_i(x) = +\infty\,, \tag{5}$$

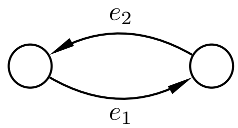

Fig. 1. The two-link network for Example 1.

then a similar proof technique as in Theorem 1 can be used to show outflow connected dynamical flow networks are always stable. In the special case when the outflow of each link only depends on the mass on the link itself, i.e., $f_i(x)$ is a function only of x_i, then condition (5) is simplified to $\lim_{x_i \to +\infty} f_i(x_i) = +\infty$. □

The condition (4) provides only a sufficient condition, since it does not distinguish on which link in the network the exogenous inflow enters. Nonetheless, there are examples of networks for which the condition is tight, as illustrated in the following example, which also illustrates that the arrival rate $\|a\|_1$ can be greater than the total exogenous inflow $\|\lambda\|_1$, and hence the need to condition the stability on the former.

Example 1: Consider the two link network depicted in Fig. 1. Suppose that $c_1 = 1$ and $c_2 \gg 1$ is large. Let $R_{1,2} = 0.8$ and $R_{2,1} = 1$. Then $a_1 = 5\lambda_1 + 5\lambda_2$ and $a_2 = 4\lambda_1 + 5\lambda_2$. If the outflow functions are of the form $f_i(x) = f_i(x_i)$ and $\liminf_{x_i \to +\infty} f_i(x_i) = 1$, the sufficient condition (4) of Theorem 1 then becomes

$$5\lambda_1 + 5\lambda_2 + \frac{4\lambda_1 + 5\lambda_2}{c_2} < 1 .$$

For the state (i.e., mass) to remain bounded, it naturally must hold that $a_1 \leq c_1$. Hence it is necessary that $\lambda_1 + \lambda_2 \leq \frac{1}{5}$, and for large c_2, the sufficient condition will become arbitrary close to a necessary condition. □

B. Necessary Condition for Stability of a Local Network

A local network is a dynamical flow network without self-loops, where all links points towards one node $v \in \mathcal{V}$, *i.e.*, $\mathcal{E}_v = \mathcal{E}$. The dynamics in (2) then simplifies to

$$\dot{x}_i = \lambda_i(t) - c_i f_i(x) , \quad \forall i \in \mathcal{E}_v . \tag{6}$$

For a local network with constant inflow, there are cases when then sufficient condition is arbitrarily close to the necessary condition, i.e., the set of inflows satisfying the necessary condition for stability is the closure of the set of flows satisfying the sufficient condition, as the following corollary shows:

Proposition 3: For a local dynamical flow network (6) that satisfy Assumption 2, if the exogenous inflows λ are constant and

$$\sum_{i \in \mathcal{E}_v} f_i(\tilde{x}) \leq \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}_v} f_i(x) , \quad \forall \tilde{x} \in \mathcal{X} ,$$

then the condition

$$\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} \leq \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}_v} f_i(x) \tag{7}$$

is necessary for stability of the local network (6).

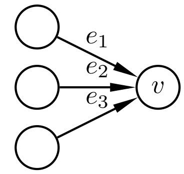

Fig. 2. The local network in Example 2. The network consists of three links, *i.e.*, $\mathcal{E}_v = \{e_1, e_2, e_3\}$.

Proof Assume that

$$\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} > \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}_v} f_i(x) .$$

Now observe that

$$x(t) = x(0) + \lambda t - C \int_0^t f(x(s)) ds .$$

Multiplying both sides by $\mathbf{1}^T C^{-1}$ we get

$$\mathbf{1}^T C^{-1}(x(t) - x(0)) = \int_0^t \left(\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} - f_i(x(s)) \right) ds ,$$

where the right hand side goes to infinity as $t \to +\infty$. Since $1^T C^{-1}$ will be a strictly positive vector and $x(t) > 0$ for all $t \geq 0$, $\sum_i x_i(t) \to +\infty$, which shows that condition (7) is necessary. ■

Next we show how Proposition 3 can be utilized to generalize previously known results.

Example 2: Consider the local network in Fig. 2 with the dynamics (6), where each link holds a queue. The service of each queue is limited such that only one queue can be served simultaneously, and there is also an overhead-time involved when switching the service between the different queues. The outflow from each queue can then be modeled as

$$f_i(x) = \frac{x_i}{\sum_{i \in \mathcal{E}_v} x_i + \kappa} , \quad \forall i \in \mathcal{E}_v , \tag{8}$$

with $\kappa > 0$ the over-head time. In [11], it is shown that

$$\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} \leq 1 . \tag{9}$$

is a necessary condition for stability for any controller when only one queue can be served simultaneously and that (9) with strict inequality is a sufficient condition for stability when using the outflow controller in (8).

Proposition 3 and Theorem 1 allow us to generalize the class of outflow functions for which this condition remains valid. In particular, we require only that $\liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}_v} f_i(x) \leq 1$ for all i, so the the stability theory is valid for outflow functions like

$$f_i(x) = (1 - e^{-\alpha_i x_i}) \frac{x_i}{\sum_i x_i + \kappa} , \quad \forall i \in \mathcal{E}_v ,$$

with $\alpha_i > 0$. □

IV. EXAMPLES

In this section, we present two different examples where the theory presented in this paper allows for a more general stability analysis than what was previously possible for these models. In the first example, we study a local queuing network with time varying inflows and non-monotone flow dynamics. In the second example, we study a multi-commodity dynamical flow network where the network topology contains cycles.

A. Queuing Networks with Time-Varying Inflows

Consider a local network with three links, $\mathcal{E}_v = \{e_1, e_2, e_3\}$, all with unit capacity $c_1 = c_2 = c_3 = 1$. Let the outflow function for each link be

$$f_i(x) = (1 - e^{-\alpha_i x_i(\sin(x_i)+1.5)}) \frac{x_i}{\sum_{j\in\mathcal{E}_v} x_j + \kappa}, \quad \forall i \in \mathcal{E}_v,$$

with $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and $\kappa = 1$. In this flow function, the first factor is a non-monotonic flow factor, while the second factor splits the service between the incoming links, just as in Example 2. Clearly,

$$\liminf_{\|x\|\to+\infty} \sum_{i\in\mathcal{E}_v} f_i(x_i) = 1. \tag{10}$$

Hence, according to Theorem 1 a sufficient condition for stability is that

$$\operatorname*{ess\,sup}_{t\geq 0} \sum_{i\in\mathcal{E}_v} \lambda_i(t) < 1.$$

For example, for inflow functions $\lambda_1(t) = 0.4 + 0.15\sin(0.05t)$, $\lambda_2(t) = 0.3 + 0.2\sin(0.05t + 2\pi/3)$, $\lambda_3(t) = 0.2 + 0.1\sin(0.05t + 4\pi/3)$, it holds that $\operatorname{ess\,sup}_{t\geq 0}\sum_{i\in\mathcal{E}_v}\lambda_i(t) = 0.99$ and hence the sufficient condition is satisfied. An example state trajectory is shown in Fig. 3(a), which also illustrates that the state converges to zero when the exogenous inflow disappear at time $t = 2000$.

To illustrate that the condition in Theorem 1 is only sufficient and not necessary for time varying inflows, we also consider the case when $\lambda_1(t) = 0.4 + 0.3\sin(0.05t)$, $\lambda_2(t) = 0.3 + 0.2\sin(0.05t)$, $\lambda_3(t) = 0.2 + 0.1\sin(0.05t)$. In this case, $\operatorname{ess\,sup}_{t\geq 0}\sum_{i\in\mathcal{E}_v}\lambda_i(t) = 1.5$, but as Fig. 3(b) shows, the state remains bounded and converges to zero when the exogenous inflow is set to zero at time $t = 2000$. In contrast, recall that, for time-invariant flows, the sufficient condition (10) becomes a necessary condition when the strict inequality is replaced with nonstrict inequality as established in Proposition 3.

B. Multi-Commodity Flows

The theory presented in this paper can easily be extended to handle multi-commodity flows. Suppose that we have two commodities, denoted A and B. Let $\lambda^A, \lambda^B \in \mathbb{R}_+^{\mathcal{E}}$ be the respective exogenous inflow of these commodities. Each commodity has its own routing matrix through the network, which we denote R^A and R^B. Both R^A and R^B are assumed to be outflow connected, i.e., satisfy Assumption 1. The state space now is the amount of mass of each commodity on

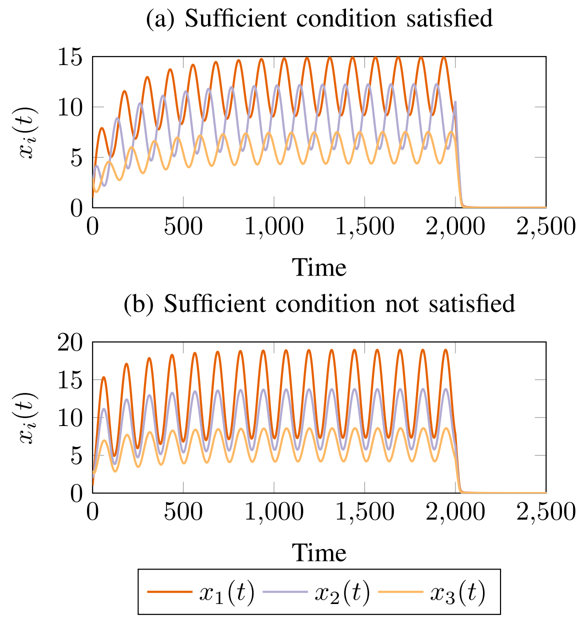

Fig. 3. The trajectories of the local queuing network in Section IV-A when: (a) the sufficient condition is satisfied; (b) when the sufficient condition is not satisfied. At $t = 2000$ the exogenous inflow is set to zero and just as expected, we can see exponential convergence to the origin. We can see that although the condition is not satisfied, the trajectories still stays bounded.

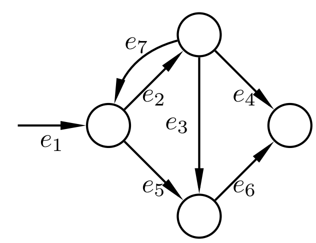

Fig. 4. The multi-flow network used for the example in Section IV-B.

every link, i.e., the state is (x^A, x^B) with $x^A, x^B \in \mathbb{R}_+^{\mathcal{E}}$. We let x denote the aggregate mass on every link, $x_i = x_i^A + x_i^B$. Under the assumption that the commodities are perfectly mixed and they move with the same aggregate flow dynamics, the flow network dynamics become

$$\dot{x}^A = \lambda^A - (I - (R^A)^T)C\operatorname{diag}\left(\left(\frac{x_i^A}{x_i}\right)_{i\in\mathcal{E}}\right) f(x),$$

$$\dot{x}^B = \lambda^B - (I - (R^B)^T)C\operatorname{diag}\left(\left(\frac{x_i^B}{x_i}\right)_{i\in\mathcal{E}}\right) f(x).$$

This model has previously been used to study traffic flows when a fleet of autonomous vehicles share the road with regular vehicles in [18]. Differently, from the results in that paper, we allow the network to have cycles.

Now, consider the network in Fig. 4. Let the outflow function for each link be $f_i(x_i) = 1 - e^{-\alpha_i x_i}$, with $\alpha_i = 1$ and the outflow capacity for each link be $c_i = 6$. The nonzero elements of the routing matrices for both commodities are

TABLE I

THE NON-ZERO ELEMENTS IN THE ROUTING MATRICES

	Commodity A	Commodity B
$R_{1,2}$	0.6	0.7
$R_{1,5}$	0.4	0.3
$R_{2,7}$	0.1	0.3
$R_{2,3}$	0.3	0.4
$R_{2,4}$	0.6	0.3
$R_{3,6}$	1	1
$R_{5,6}$	1	1
$R_{7,2}$	0.5	0.3
$R_{7,5}$	0.5	0.7

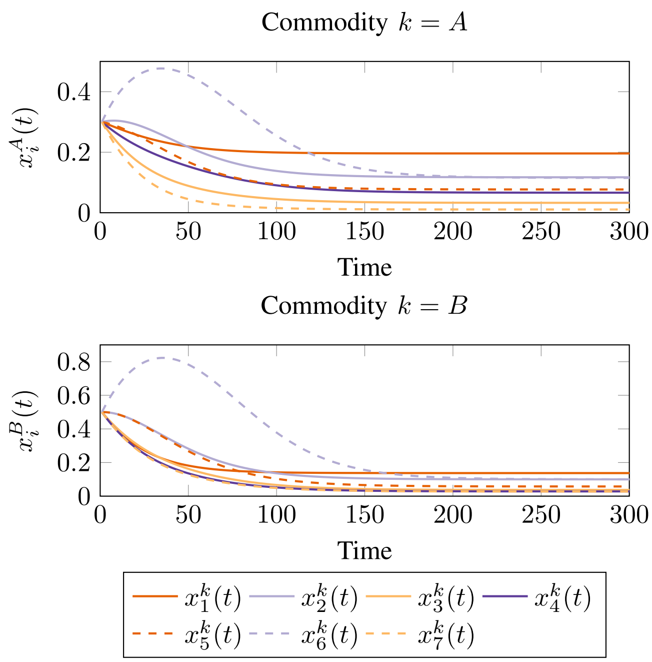

Fig. 5. The trajectories for commodity A and commodity B respectively in the example in Section IV-B.

given in Table I. Define

$$a^A(t) = (I - (R^A)^T)^{-1}\lambda^A(t)\,,$$
$$a^B(t) = (I - (R^B)^T)^{-1}\lambda^B(t)\,.$$

By using the Lyapunov function

$$V(x) = (I - (R^A)^T)^{-1}x^A + (I - (R^B)^T)^{-1}x^B\,,$$

and the same theory as in the proof of Theorem 1, we obtain the following sufficient condition for stability of the multi-commodity dynamics

$$\operatorname*{ess\,sup}_{t\geq 0} \sum_{i\in\mathcal{E}} \frac{a_i^A(t) + a_i^B(t)}{c_i} < \liminf_{\|x\|\to+\infty} \sum_{i\in\mathcal{E}} f_i(x)\,. \qquad (11)$$

For this specific example, $\liminf_{\|x\|\to+\infty}\sum_{i\in\mathcal{E}} f_i(x_i) = 1$. By letting $\lambda_1^A = 1$, $\lambda_1^B = 0.7$ and all other elements of λ^A, λ^B be zero, the sufficient condition in (11) is satisfied. The trajectory for each commodity is shown in Fig. 5, with the initial states $x_i^A(0) = 0.3$ and $x_i^B(0) = 0.5$ for all $i \in \mathcal{E}$.

V. CONCLUSIONS

In this paper, we have shown how the Strong integral Input-to-State Stability (Strong iISS) provides conditions for the stability of dynamical flow networks. We established sufficient conditions on the exogenous inflows for dynamical flow networks to be stable and showed that the condition is also necessary for local networks under certain assumptions and for certain types of networks. We also showed how the conditions can be applied to existing dynamical flow network models and provided stability assurance in settings not covered in prior literature.

A future research direction is to explore if the theory of Strong iISS leads to alternative tighter bounds in certain settings, e.g., by considering the time-averaged exogenous inflow or dividing the network into several sub-networks. The latter will provide a way to overcome the conservatism that the Strong iISS property needs for its generalizability.

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