Lyapunov Differential Equation Hierarchy and Polynomial Lyapunov Functions for Switched Linear Systems

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Abstract— **This work studies the problem of searching for homogeneous polynomial Lyapunov functions for stable switched linear systems. Specifically, we show an equivalence between polynomial Lyapunov functions for systems of this class and quadratic Lyapunov functions for a related hierarchy of Lyapunov differential equations. This creates an intuitive procedure for checking the stability properties of switched linear systems, and a computationally competitive algorithm is presented for generating high-order homogeneous polynomial Lyapunov functions in this manner. Additionally, we provide a comparison between polynomial Lyapunov functions generated with our proposed approach and Lyapunov functions generated with a more traditional sum-of-squares based approach.**

I. Introduction

The structure of switched dynamical system models has been widely explored and exploited in order to the analyze stability and performance of real-world systems [1], [2]. In turn, such results have inspired the use of switched systems as a modeling tool for many challenging analysis problems. For example, hybrid dynamical systems can be represented as switched systems, as can some stochastic systems [3], [4]. Certain nonlinearities such as saturation and mechanical backlash can be modeled using switched linear systems [1], [5]–[7], as can random noise [8]. Switched linear systems can be used as an over-approximating abstraction for more general nonlinearities [5], [9] and, for this reason, switched linear system models appear widely in robustness analysis literature [6], [10].

Stability-type proofs for switched dynamical systems often require the construction of polynomial Lyapunov functions. The simplest class of polynomial Lyapunov function is the class of *quadratic* Lyapunov functions and, as such, the search for quadratic Lyapunov functions has computational advantages in comparison to other methods of stability analysis. Numerous works, including [7], [11] explore the guarantees attainable when solely searching for quadratic Lyapunov functions, however, recent progress in sum-of-squares based techniques have shown that higher-order polynomial Lyapunov functions can be calculated as well with more accurate stability guarantees [12], [13]. In general, sum-of-squares based techniques require little machinery to implement; these methods cast the search for a polynomial Lyapunov function as a convex feasibility problem, and many efficient solvers exist to solve such problems [12]. Additionally, for systems which are known to be stable, the computation of high-order polynomial Lyapunov functions has the ability to help characterize invariant regions of the state space with complex geometries; this is not possible when computing quadratic Lyapunov functions.

This work provides an algorithm, posed as a convex feasibility problem, for constructing homogeneous polynomial Lyapunov functions for switched linear systems. Importantly, the structure of our algorithm differs significantly from traditional sum-of-squares formulations, as we encode the search for polynomial Lyapunov functions as a search for quadratic Lyapunov functions for a related hierarchy of Lyapunov differential equations. This creates an intuitive procedure for checking the stability properties of switched linear systems and enables new applications as well [8]. Moreover, we show that every homogeneous sum-of-squares polynomial Lyapunov function for a given initial system can be transformed to a quadratic polynomial Lyapunov function for a system in the related hierarchy; this procedure can also be conducted in the reverse order, allowing one to generate sum-of-squares polynomial Lyapunov functions for an initial system through the identification of a quadratic polynomial Lyapunov function for a related system.

This paper is organized in the following way. We introduce the time-varying Lyapunov differential equation in Section II, which we define in reference to an initial switched linear system. Using the time-varying Lyapunov differential equation as an initial case, we then form a hierarchy of Lyapunov differential equations in Section III, and quadratic Lyapunov functions for differential equations in this hierarchy are shown to correspond to homogeneous polynomial Lyapunov functions for the initial switched system. Section IV explores the relation between quadratic Lyapunov functions for the aforementioned hierarchy of Lyapunov differential equations and homogeneous sum-of-squares polynomial Lyapunov functions for the initial switched linear system. Finally, we provide an algorithm for computing high-order homogeneous polynomial Lyapunov functions for switched linear systems in Section V; this algorithm is presented in conjunction with a numerical example.

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II. Stability and Switched Linear Systems

Consider the linear time-variant system

$$\dot{x} = A(t)x, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, and $A(t) \in \mathbb{R}^{n\times n}$ evolves nondeterministically inside a finite set of switched linear modes $A(t) \in \{A_1, \cdots, A_N\}$. We assume that each switched mode $\dot{x} = A_i x$, with $i \in \{1, \cdots, N\}$, is asymptotically stable.

In this work, we study common polynomial Lyapunov functions for systems of the form (1). A common Lyapunov function is a mapping $V : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, with $x \neq 0$, and for $i \in \{1, \cdots, N\}$ we have

$$V(x) > 0 \quad \text{and} \quad \dot{V}(x) = \langle \nabla V, A_i x\rangle < 0, \tag{2}$$

and it is well known that the system (1) is stable if and only if there exists a V satisfying (2). Moreover, the authors of [14] show that (1) is stable if and only if there exists a *homogeneous* polynomial Lyapunov function satisfying (2). We capture this assertion in Remark 1.

Remark 1. [15, Theorem 4.5] If the switched linear system (1) is asymptotically stable under arbitrary switching, then there exists a polynomial Lyapunov function $V(x)$, satisfying (2), which is homogeneous in the entries of x. ■

In the special instance that there exists a $V(x)$, satisfying (2), which is quadratic in the entries of x, we say that the system (1) is *quadratically stable* [7]. Such a Lyapunov function will take the form $V(x) = x^T P x$ where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and

$$A_i^T P + P A_i \; < \; 0 \tag{3}$$

for all $i \in \{1, \cdots, N\}$.

Quadratic polynomial Lyapunov functions are the simplest substantiation of homogeneous polynomial Lyapunov functions, and thus, the search for a quadratic Lyapunov function for (1) has computational advantages in comparison to other strategies for stability analysis; the search can be reduced to solving a convex feasibility problem involving linear matrix inequalities, and many efficient solvers exist to solve such problems [6], [16]. Recent progress in polynomial optimization systems via sum-of-squares relaxations, however, has shown that more general polynomial Lyapunov functions could be computed as well with added benefits, such as improved system stability margins.

Importantly, if the system (1) is linear time-invariant, i.e. $N = 1$, then (1) is asymptotically stable if and only if there exists a P satisfying (3). This is not true, however, in the case of multiple switched modes; stable switched linear systems exist for which there is no quadratic Lyapunov function certifying the stability of each mode [11, Section 3]. For this reason, we must resort to more complex tools to prove stability in the general setting of (1).

We next present the time-variant switched Lyapunov differential equation:

$$\dot{X} = A(t)X + XA(t)^T, \tag{4}$$

where $X(t) \in \mathbb{R}^{n\times n}$ and $A(t)$ retains its definition from (1). Importantly, stability guarantees on the Lyapunov system (4) propagate down to stability guarantees on the initial system.

Proposition 1. *The switched Lyapunov differential equation* (4) *is stable if and only if the system* (1) *is stable.* ■

III. Establishing a Hierarchy of Lyapunov Differential Equations

In this section we build on (4) to create a hierarchy of Lyapunov differential equations for the system (1). As was the case in Proposition 1, each system in the hierarchy is shown to have equivalent stability properties.

A. Notation

Let $A \otimes B \in \mathbb{R}^{np\times mq}$ denote the Kronecker product of $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$. Let $\otimes^k x \in \mathbb{R}^{n^k}$ denote the k^{th} Kronecker power of $x \in \mathbb{R}^n$, which is defined recursively by

$$\begin{aligned} \otimes^1 x &= x && \in \mathbb{R}^n, \\ \otimes^k x &= x \otimes (\otimes^{k-1} x) && \in \mathbb{R}^{n^k}, \quad k \geq 2. \end{aligned}$$

Let $W^+ \in \mathbb{R}^{m\times n}$ denote the Moore-Penrose inverse of $W \in \mathbb{R}^{n\times m}$, and let $I_n \in \mathbb{R}^{n\times n}$ denote the $n \times n$ identity matrix.

B. Identifying Meta-Lyapunov Functions

We first rewrite (4) as

$$\dot{\vec{X}} = \mathcal{A}(t)\vec{X} \tag{5}$$

by taking $\vec{X}$ to be the vectorization of X, i.e. $\vec{X} = \text{vec}(X) \in \mathbb{R}^{n^2}$. In this case, $\mathcal{A}(t) \in \mathbb{R}^{n^2\times n^2}$ evolves nondeterministically in the set $\mathcal{A}(t) \in \{\mathcal{A}_1, \cdots, \mathcal{A}_N\}$ where, for $i \in \{1, \cdots, N\}$, we define $\mathcal{A}_i := I_n \otimes A_i + A_i \otimes I_n$.

We refer to (5), which is also linear time-variant, as the *meta-system* relative to system (1). Applying concepts of quadratic stability to meta-systems, the system (5) is stable if there exists a positive definite $P \in \mathbb{R}^{n^2\times n^2}$ such that

$$\mathcal{A}_i^T P + P\mathcal{A}_i < 0, \tag{6}$$

for all $i \in \{1, \cdots, N\}$. These constraints correspond to the existence of a Lyapunov function $\mathcal{V}(\vec{X}) = \vec{X}^T P \vec{X}$ for (5), which is quadratic in the entries of $\vec{X}$. In what follows, we refer to $\mathcal{V}(\vec{X})$ as a *meta-Lyapunov function* for the system (1).

Theorem 1. *If the system* (1) *is quadratically stable, then the system* (5) *is also quadratically stable.*

It is of course possible to repeat the process again and certify stability at a deeper level; for instance, one may form the Lyapunov differential equation corresponding to (5),

$$\dot{\xi} = (I \otimes \mathcal{A}(t) + \mathcal{A}(t) \otimes I)\xi, \tag{7}$$

$\xi \in \mathbb{R}^{n^4}$ and then show that the system (5) is quadratically stable if (7) is quadratically stable. Pursuing the process further, it is possible to construct a "hierarchy" of Lyapunov differential equations whose state space dimensions are n^{2^c}, where c is an integer greater than or equal to 1. In the following section, we complete this hierarchy to include Lyapunov differential equations whose state space dimensions are n^{2c}.

C. A Linear Hierarchy of Polynomial Lyapunov Functions

We next develop a hierarchy of dynamical systems whose state space dimensions grow as integer exponents of n, the dimension of the state space of (1). This hierarchy complements the hierarchy of systems discussed above.

Theorem 2. *System* (1) *is stable if there exists* $c \in \mathbb{N}_{\geq 1}$ *and* $P_c \in \mathbb{R}^{n^c \times n^c}$ *positive definite such that*

$$\mathcal{A}_{c,i}^T P_c + P_c \mathcal{A}_{c,i} < 0 \tag{8}$$

for all $i \in \{1, \cdots, N\}$*, where*

$$\mathcal{A}_{c,i} := \sum_{j=0}^{c-1} I_{n^j} \otimes A_i \otimes I_{n^{c-1-j}}. \tag{9}$$

Proof. Taking $\vec{X} = \otimes^c x(t) \in \mathbb{R}^{n^c}$, we find

$$\dot{\vec{X}} = \mathcal{A}_c(t)\vec{X} \tag{10}$$

where $\mathcal{A}_c$ is given by (8), and the stability of (10) implies that of (1). Therefore the system (1) is stable if there exists a positive definite $P_c \in \mathbb{R}^{n^c \times n^c}$ such that (8) holds. □

Theorem 2 shows that the existence of a $P_c \in \mathbb{R}^{n^c \times n^c}$ satisfying (8) for some integer $c \geq 1$ certifies the stability of (1); such a P_c identifies

$$V_c(x) = (\otimes^c x(t)^T) P_c (\otimes^c x(t)) \tag{11}$$

as a polynomial Lyapunov function for (1), which is homogeneous in the entries of x and of order $2c$. Importantly, the degree of $V_c(x)$ grows linearly with c.

D. Reducing the Dimensionality of the Meta-System

The benefits of searching for meta-Lyapunov functions for (1) using the methods presented thus far are namely structural; (8)-(9) provide an intuitive procedure for generating high-order homogeneous polynomial Lyapunov function for (1) and moreover, this procedure does not require any heavy machinery to implement. In contrast, there are few computational advantages to this approach, at present. This is due in part to internal redundancy built into the Lyapunov constraints given by (8). We demonstrate this assertion through the following example.

Example 1. Consider the system (1) evolving in $\mathbb{R}^2$. In this case, $x = [x_1,\, x_2]^T \in \mathbb{R}^2$, and $\vec{X} \in \mathbb{R}^4$ is given by

$$\vec{X} = x \otimes x = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_2 & x_2^2 \end{bmatrix}^T.$$

When beginning at an initial condition $\vec{X}(0) = x(0) \otimes x(0)$ and evolving along trajectories of the meta-system

$$\dot{\vec{X}} = (I_2 \otimes A(t) + A(t) \otimes I_2)\vec{X}, \tag{12}$$

we find that the second and third entries of $\vec{X}$ remain equal to one another, regardless of the switching policy. This is due to the construction of $(I_2 \otimes A(t) + A(t) \otimes I_2)$. The methods presented thus far address the problem of searching for a meta-Lyapunov function $\mathcal{V}(\vec{X}) = \vec{X}^T P \vec{X}$ for the system (12), however, we now see that that the constraints on P, given by (8), contain internal redundancy.

Now, consider a vector containing the second-order monomials of x, this time with no redundancy. Specifically, consider $y(x) = [x_1^2,\, x_1x_2,\, x_2^2] \in \mathbb{R}^3$, and note that $\vec{X} = Wy(x)$ where

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From (10), we have that $\dot{y} = W^+ \mathcal{A}_c(t) W y$ and, thus, one can now formulate the search for a fourth-order homogeneous polynomial Lyapunov function for (1), as the search for a quadratic Lyapunov function $\overline{V}(y) = y^T \overline{P} y$ that certifies the stability of y. ■

As shown in the previous example, the constraints given by (8) are redundant; that is, a quadratic Lyapunov function that certifies the stability of $\vec{X}$, as in (10), will individually certify the stability of each of the meta-system's states, whereas, a reduced order meta-Lyapunov function that stabilizes a subset of meta-system's states may be sufficient.

For this reason, we present a new formulation of the constraints (8)-(9) that contains no redundancy. We begin with the following definition.

Definition 1 ($\mathcal{A}_c$-Invariant Subspaces)**.** A subspace $S \subset \mathbb{R}^n$ is $\mathcal{A}_c$*-invariant* for (10) if for every vector $v \in S$ and every matrix $\mathcal{A}_{c,i}$ with $i \in \{1, \cdots, N\}$ we have $\mathcal{A}_{c,\,i} v \in S$. ■

Note that $\mathcal{A}_c(t)$ as in (10) will have an inherent invariant subspace resulting from its construction. We next remove this redundancy and analyze a reduced order meta-system whose states correspond to unique monomials of the initial switched system (1). While the initial meta-Lyapunov conditions (8) are defined by n^{2c} constraints per switched mode, our new formulation only requires $M(n,\, c)^2$ such constraints, where $M(n,\, c)$ denotes the number of monomials of order $c \in \mathbb{N}_{\geq 1}$ in the entries of $x \in \mathbb{R}^n$ and is given by

$$M(n,\, c) = \binom{c+n-1}{n-1}.$$

This result is encapsulated in the following theorem.

Theorem 3. *Let* $y_c(x) \in \mathbb{R}^{M(n,\, c)}$ *denote a vector containing the monomials of* x *of order* c*, which we define in conjunction with a matrix* $W_c \in \mathbb{R}^{n \times M(n,\, c)}$*:*

$$\otimes^c x = W_c y_c(x). \tag{13}$$

Additionally, let $\overline{P}_c \in R^{M(n,\, c) \times M(n,\, c)}$ *be symmetric positive definite. If*

$$B_{c,\,i}^T \overline{P}_c + \overline{P}_c B_{c,\,i} < 0 \tag{14}$$

for all $i \in \{1, \cdots,\ N\}$ *where*

$$B_{c,\,i} := W_c^+ \mathcal{A}_{c,\,i} W_c, \tag{15}$$

then $V_c(x) = y_c(x)^T\, \overline{P}\, y_c(x)$ *is a homogeneous polynomial Lyapunov function for* (1) *of order* $2c$*.*

The proof of this result comes from the fact that $A_c(t)$ has an inherent invariant subspace, resulting from its construction. As trajectories of (10) are known to begin in this subspace, we can encode the search for meta-Lyapunov functions for the system (1) as a search for quadratic Lyapunov functions for the reduced order system

$$\dot{y}_c = B_c(t) y_c \tag{16}$$

where $B_c(t) \in \{B_{c,1}, \cdots, B_{c,N}\}$ and $y(x)$ is given by (13).

Importantly, Theorem 3 allows the system designer to select $y_c(x)$ with whatever ordering properties they like; that is, we do not assume an order to the monomials that are stored in $y_c(x)$. However, each ordering will induce a unique W_c, and thus the resulting Lyapunov conditions will always be the same, regardless of the chosen ordering. In the specific case where $n = 2$, there is an intuitive ordering to the monomials of x; under this assumed ordering, the matrix W_c, given by (13), can be captured in closed form.

Proposition 2. *Consider the system* (1) *and let* $n = 2$. *In this case we have* $M(n, c) = c + 1$. *For a positive integer* $k \in \mathbb{N}_{\geq 0}$*, let* $0_k \in \mathbb{R}^k$ *denote a vector populated with zeros.*

If $y_c(x) = [x_1^c,\, x_1^{c-1}x_2,\, \cdots,\, x_2^c]^T \in \mathbb{R}^{c+1}$ *then we have* $\otimes^c x = W_c\, y_c(x)$*, where for an integer* $k \in \mathbb{N}_{\geq 1}$ *we define* W_k *recursively by*

$$W_k = \left[\begin{array}{cc} W_{k-1} & 0_{2^{k-1}} \\ \hline 0_{2^{k-1}} & W_{k-1} \end{array}\right] \quad k \geq 2. \tag{17}$$

for $W_1 = I_2$. ■

In the case when $n > 2$, it is generally difficult to order the c^{th} order monomials of x in an intuitive way. For this reason, we do not expand Proposition 2 to account for the case where $n > 2$, nor do we suggest a canonical ordering for the entries of $y_c(x)$. However, W_c can always be solved for using (13) once $y_c(x)$ has been chosen.

IV. Relation to Homogeneous Polynomial Lyapunov Functions

Traditionally, the search for a polynomial Lyapunov functions systems of the form (1) is encoded as the search for a sum-of-squares polynomial $V(x)$, satisfying (2).

Definition 2. A polynomial $p(x)$ is a *sum-of-squares* in x if there exist polynomials $g_1, \cdots, g_r : \mathbb{R}^n \to \mathbb{R}$ such that $p(x) = \sum_{i=1}^r g_i(x)^2$. ■

The search for a sum-of-squares polynomial $V(x)$, satisfying (2), is known to be a convex optimization problem, computable by solving a semidefinite program [13]. Many efficient solvers exist to handle such problems [6], [16].

We next show that the existence of quadratic Lyapunov functions for the hierarchy of dynamical systems (10) guarantees the existence of a homogeneous sum-of-squares polynomial Lyapunov functions for (1), and vice versa. In this sense, all homogeneous sum-of-squares polynomial Lyapunov functions can be thought of as quadratic Lyapunov functions for a related hierarchy of differential equations. Moreover, one can encode the search for high-order sum-of-squares polynomial Lyapunov functions for (1) as a search for quadratic Lyapunov functions for (10).

Theorem 4. *There exists a* $\overline{P}_c \in \mathbb{R}^{M(n,\,c)\times M(n,\,c)}$ *satisfying* (14) *for some positive integer* $c \in \mathbb{N}_{\geq 1}$*, if and only if there exists a homogeneous sum-of-squares polynomial Lyapunov function* $V_c(x)$ *of degree* $2c$ *for the system* (1).

Proof. A sum-of-squares polynomial that is homogeneous in the entries x and of order $2c$ will take the form $p(x) = y_c(x)^T Z y_c(x)$, where $Z \in \mathbb{R}^{M(n,c)\times M(n,c)}$ is symmetric, and $y_c(x)$ and $M(n, c)$ retain their definitions from Theorem 3. From Theorem 3, we have that if $\overline{P}_c \in \mathbb{R}^{M(n,\,c)\times M(n,\,c)}$ satisfies (8) for some positive integer $c \in \mathbb{N}_{\geq 1}$, then we have that $V_c(x) = y_c(x)^T \overline{P} y_c(x)$ is a homogeneous polynomial Lyapunov function for (18) and, moreover, $V_c(x)$ is a sum-of-squares. To prove the converse, we note that if $p(x) = y_c(x)^T Z y_c(x)$ is a homogeneous sum-of-squares polynomial Lyapunov function for (18) then $Z > 0$ and $\dot{p}(x) < 0$ for all $x \in \mathbb{R}^n$. From the dynamics of $y_c(x)$, given as (16), we have $B_{c,\,i}^T Z + Z B_{c,i} < 0$ for all $i \in \{1, \cdots, N\}$. Therefore $\overline{P}_c = Z$ solves (14). □

V. Numerical Example

We now provide an example case and prove the stability of a switched linear system using a meta-Lyapunov function based approach. An algorithm is provided for generating homogeneous polynomial Lyapunov functions for switched systems, which follows the procedure detailed in Theorem 3; this algorithm is specifically written for implementation with CVX, a convex optimization toolbox made for use with MATLAB [17]. We also provide a comparison to a similar search for homogeneous polynomial Lyapunov functions that was implemented using SOSTOOLS, a sum-of-squares optimization toolbox made for use with MATLAB [18]. Experimental results are provided from MATLAB 2019b, which was run on a 2017 Macbook Pro laptop.

A. Problem Formulation

We consider the stable linear time-variant system

$$\dot{x} = A(t)x \qquad A(t) \in \{A_1,\, A_2\} \tag{18}$$

$$A_1 = \begin{bmatrix} -.5 & .5 \\ -.5 & -.5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -2.5 & 2.5 \\ -2.5 & 1.5 \end{bmatrix}.$$

In the following, we compute homogeneous polynomial Lyapunov functions for (18) using Theorem 3.

Importantly, if the system (18) begins at an initial position $x_0 = x(0)$ and $V_c(x)$ satisfies (2), then for all $t \geq 0$ we have

$$x(t) \in \{x \in \mathbb{R}^n \,|\, V_c(x) \leq V_c(x_0)\}. \tag{19}$$

For this reason, we select $V_c(x)$ as the minimizers of a suitable objective function, as to shrink the resulting invariant region derived through (19). In what follows, we show that computing higher-order meta-Lyapunov functions allows one to characterise tighter invariant sets by (19), even when the same objective function is used in each computation.

Algorithm 1 Computing Meta-Lyapunov Functions

input : $A_1,\ A_2 \in \mathbb{R}^{2\times 2}$ from (1). $c \in \mathbb{N}_{\geq 1}$.
output: $\overline{P}_c \in \mathbb{R}^{(c+1)\times(c+1)}$ satisfying (14).

1: **function** METALYAPUNOV($A_1,\ A_2,\ c$)
2:     **Initialize:** Compute $\mathcal{A}_{c,1}$ and $\mathcal{A}_{c,2}$ by (9)
3:         Compute W_c by (17)
4:     $B_{c,1} \leftarrow W_c^{+}\mathcal{A}_{c,1}W_c$
5:     $B_{c,2} \leftarrow W_c^{+}\mathcal{A}_{c,2}W_c$
6:     **cvx_begin sdp**
7:     **variable** $\overline{P}_c(c+1, c+1)$ **semidefinite**
8:     $0\ > B_{c,1}^T\,\overline{P}_c + \overline{P}_c\,B_{c,1}$
9:     $0\ > B_{c,2}^T\,\overline{P}_c + \overline{P}_c\,B_{c,2}$
10:     $\overline{P}_c > I_n$
11:     %% Possibly Insert Objective Function
12:     **cvx_end**
13:     **if** Program feasible **then**
14:         **return** $\overline{P}_c$
15: **end function**

B. Identifying Meta-Lyapunov Functions

We search for meta-Lyapunov functions for (18) using a semidefinite program. Specifically, when searching for a homogeneous Lyapunov function of order $2c$, we first calculate $B_{c,1}$ and $B_{c,2}$ using (9), (15) and (17), and then we search for a $\overline{P}_c$ satisfying (14). Such a matrix identifies $V_c(x) = y_c(x)^T\,\overline{P}_c\,y_c(x)$ as a polynomial Lyapunov function for (18), which is homogeneous in the entries of x and of order $2c$. We implement the aforementioned procedure with Algorithm 1, which specifically relies on CVX, a convex optimization toolbox built for use with MATLAB [17], [19]. Algorithm 1 takes as inputs the system parameters A_1 and A_2, and a positive integer c, and returns a matrix $\overline{P}_c$, in the case that one exists, which satisfies (8) at the c^{th} level.

Note that Algorithm 1 computes a feasibility problem, rather than an optimization problem; that is, while Algorithm 1 searches for a $\overline{P}_c$ that satisfies the meta-Lyapunov constraint (14), this solution is computed without referencing any objective function. Note however, that in the instance that multiple feasible solutions exist, it is preferable to choose $\overline{P}_c$ such that the sublevel sets of the resulting homogeneous Lyapunov function $V_c(x) = y_c(x)^T\,\overline{P}_c\,y_c(x)$ are small. For this reason, it is desirable to compute $\overline{P}_c$ as the solution to an optimisation problem, rather than a feasibility problem.

Little is known, in general about how one can relate the parameters of a polynomial to the volume of its sublevel sets. In our case as well, it is difficult to associate a metric of optimality with solutions to (14). Through experimentation, we have generally found that it is preferable to use either the objective function

11: **minimize** $\overline{P}_c(1,\ 1)$

which minimises the coefficient on x_1^{2c} in the resultant Lyapunov function $V_c(x)$, or

11: **minimize** $\overline{P}_c(c+1,\ c+1)$

which minimises the coefficient on x_2^{2c}. These objective functions are provided in psuedocode, such that they can easily be inserted in Algorithm 1 at line 11.

C. Numerical Results and Comparison with SOSTOOLS

We now return to the example system (18), and compute feasible meta-Lyapunov functions with Algorithm 1. Additionally, we compute an over approximation of the infinite time reachable set of (18) when beginning from the initial conditions $x(0) = [1,\ 0]^T$. As suggested in the preceding, we compute these invariant sets by solving (14), while attempting to minimize $\overline{P}(1, 1)$; see Algorithm 1, Line 11. This procedure was computed in MATLAB 2019b using CVX.

In the case of this example, Algorithm 1 was computed for $c \in \{1, 2, \cdots, 13\}$, thus generating homogeneous polynomial Lyapunov functions for all even orders between 2 and 26. These Lyapunov functions were then used to calculate invariant regions of the state space using (19); see Figure 1[1]. Note that as the order of the meta-Lyapunov function increases, the derived invariant sets shrink in volume. Further, certain higher-order the meta-Lyapunov functions were shown to have non-convex sublevel sets. We provide the number of solver iterations for each experiment, as well as the computations times, in Figure 2.

We next compare Algorithm 1 to a more traditional sum-of-squares based search for homogeneous polynomial Lyapunov functions. Specifically, we implement SOSTOOLS [18] with the solver SDPT3 and attempt to generate homogeneous polynomial Lyapunov functions for the system (18) while minimizing the coefficient on x_1^{2c}. We provide the number of solver iterations for each experiment, as well as the computations times in Figure 2.

In the experiment, SOSTOOLS was only able to generate homogeneous polynomial Lyapunov functions of order 20 or below; a solver error was returned during each search for more complex Lyapunov function. In contrast, Algorithm 1, when implemented through CVX, was able to generate up to 26^{th}-order polynomial Lyapunov functions while minimizing the same objective function. We attribute this discrepancy to the fact that many of the steps required in a traditional sum-of-squares based search optimization are not required by Algorithm 1. For example, Algorithm 1 does not compute the time rate of change of the monomials in x of order $2c$; that is, Algorithm 1 begins with a closed-form representation of $\dot{y}_c(x)$, which is encoded in the matrices $B_{c,1}$ and $B_{c,2}$. SOSTOOLS must compute $\dot{y}_c(x)$ online as a sum of squares of lower-order monomials and, for this reason, one can expect SOSTOOLS to perform less efficiently.

Despite this, in the experiments where SOSTOOLS was able to correctly generate homogeneous Lyapunov functions,

[1]The code for this experiment is publicly available though the GeorgiaTech Facts Lab Github: https://github.com/gtfactslab/Abate_ACC2020.

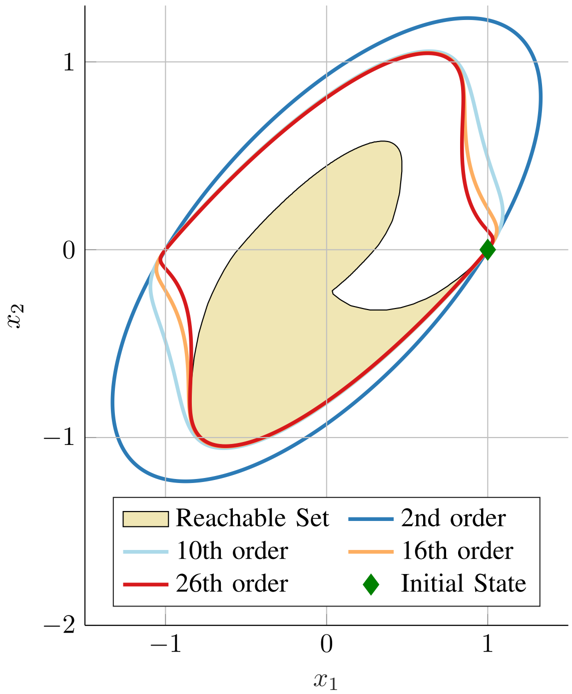

Fig. 1: Simulated system response of (18). When starting from $x_0 = [1,\ 0]^T$, the system can only reach the region shown in light yellow, which was computed via simulation. The dark blue, light blue, orange and red regions represent invariant sets calculated using quadratic, 10^{th}-order, 16^{th}-order and 26^{th}-order meta-Lyapunov functions, respectively. The invariance of these regions is shown by (19).

2c	MetaLyapunov		SOSTOOLS	
	Iterations	Time (s)	Iterations	Time (s)
2	9	1.51	11	0.40
10	20	1.68	25	0.56
20	36	2.40	33	0.95
24	40	2.48	31	N/A
26	54	3.30	31	N/A
28	36	N/A	31	N/A

Fig. 2: Comparing Algorithm 1 to a search for homogeneous Lyapunov functions using SOSTOOLS. Algorithm 1 is used to compute homogeneous Lyapunov functions of orders 2, 10, 20, 24, and 26 for the system (18). SOSTOOLS, however, is only able to correctly generate Lyapunov functions of order 20 and below. The computation time and number of solver iterations are provided for each experiment. The symbol N/A is used when the solver fails; in this case, we provide the number of solver iterations performed prior to failure.

we found that SOSTOOLS performed faster in computation that the meta-Lyapunov search implemented with CVX; however, it did take SDPT3 more solver iterations to generate the solution which minimized the objective function when implemented through SOSTOOLS (Figure 2).

VI. Conclusion

This work addresses the problem of searching for homogeneous polynomial Lyapunov functions for stable switched linear systems. An equivalence is shown between polynomial Lyapunov functions for switched linear systems and quadratic Lyapunov functions for a related hierarchy of Lyapunov differential equations. A computationally competitive algorithm is presented for generating high-order homogeneous polynomial Lyapunov functions in this manner.

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