Improving the Fidelity of Mixed-Monotone Reachable Set Approximations via State Transformations

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***Abstract*— Mixed-monotone systems are separable via a decomposition function into increasing and decreasing components, and this decomposition function allows for embedding the system dynamics in a higher-order monotone embedding system. Embedding the system dynamics in this way facilitates the efficient over-approximation of reachable sets with hyperrectangles, however, unlike the monotonicity property, which can be applied to compute, *e.g.*, the *tightest* hyperrectangle containing a reachable set, the application of the mixed-monotonicity property generally results in conservative reachable set approximations. In this work, we explore conservatism in the method, and we consider, in particular, embedding systems that are monotone with respect to an alternative partial order. This alternate embedding system is constructed with a decomposition function for a related system formed via a linear transformation of the initial state-space. We show how these alternate embedding systems allow for computing reachable sets with improved fidelity, *i.e.*, reduced conservatism.**

I. Introduction

A dynamical system is mixed-monotone if there exists a related *decomposition function* that decomposes the system's vector field into increasing and decreasing components; mixed-monotonicity applies to continuous-time systems [1]–[4], discrete-time systems [5], as well as systems with disturbances [6]–[8], and it generalizes the *monotonicity* property of dynamical systems for which trajectories maintain a partial order over states [9], [10].

For an n-dimensional mixed-monotone system with a disturbance input, it is possible to construct a $2n$-dimensional monotone *embedding system* from the decomposition function, and this embedding system contains no disturbances. Thus, tools from monotone systems theory can be applied to the embedding system to conclude properties of the original dynamics; in particular, such approaches are useful to efficiently approximate reachable sets using hyperrectangles. For example, it is shown in [6]–[8] how finite-time forward reachable sets for the original system are efficiently approximated via a single simulation of the embedding system, and this procedure is extended in [11] for the approximation of backward-time reachable sets. These works assume a hyperrectangular initial set of interest, and the approximations derived from their procedures are also hyperrectangles.

This work was supported in part by the National Science Foundation under grant #1749357 and the Air Force Office of Scientific Research under award FA9550-19-1-0015.

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Unlike the monotonicity property, which can be applied to compute the *tightest* hyperrectangle containing a reachable set [9], the application of the mixed-monotonicity property is known to generally result in conservative reachable set approximations. In this work, we explore two main ways of reducing the conservatism in the approach: (i) using alternative and/or multiple decomposition functions, and (ii) using alternative and/or multiple partial orders.

The first topic was recently explored in [12], [13] and it is now known that all systems are mixed-monotone with a unique *tight* decomposition function that computes reachable sets with less conservatism than any other decomposition function. This tight construction is defined as an optimization problem and, for this reason, it can be preferable to employ a different construction; see [6], [7], [14] for generating decomposition functions from bounds on the systems Jacobian matrices. Our first result is to show how two initial decomposition functions for a given system can be combined in a piecewise fashion to create a new decomposition function for the same system that approximates reachable sets with accuracy at least as good as both initial decomposition functions. This method for reducing conservatism is particularly useful when both initial decomposition functions are derived using the Jacobian bound approach from [6], [7], [14], as this approach can, in some instances, produce multiple distinct decomposition functions for the same system.

The main results of this paper, however, deal with the second topic regarding alternative partial orders. In particular, we consider the standard componentwise partial orders in a linearly transformed state-space, and we observe that inequality intervals in the transformed space correspond to parallelotopes in the original state-space. Thus, it is possible to compute parallelotope over-approximations of reachable sets by applying the standard mixed-monotonicity tools with the new order. We present two methods for reducing conservatism in this manner: (i) several different partial orders can be used so that the reachable set of the system is known to lie in the intersection of the approximation derived from each partial order, (ii) in certain cases, a linear transformation can be found to transform the system to a monotone system. As a tight decomposition function is known to exist for any given transformed system, there exists an analogous notion of tightness with respect to any given parallelotope shape [1].

[1]The code that generates the figures in this work is publicly available through the GeorgiaTech FactsLab GitHub: `https://github.com/gtfactslab/Abate_ACC2021_2`. Certain proofs in this work are omitted and appear in an extended version of this work available through arXiv: `https://arxiv.org/abs/2010.01065`.

II. NOTATION

Let (x, y) denote the vector concatenation of $x, y \in \mathbb{R}^n$, *i.e.*, $(x, y) := [x^T\, y^T]^T \in \mathbb{R}^{2n}$, and let $\preceq$ denote the componentwise vector order, *i.e.*, $x \preceq y$ if and only if $x_i \leq y_i$ for all $i \in \{1, \cdots, n\}$ where vector components are indexed via subscript. Given $x, y \in \mathbb{R}^n$ with $x \preceq y$,

$$[x, y] := \{z \in \mathbb{R}^n \mid x \preceq z \text{ and } z \preceq y\} \quad (1)$$

denotes the hyperrectangle defined by the endpoints x and y, and given a nonsingular transformation matrix $T \in \mathbb{R}^{n\times n}$,

$$[x, y]_T := \left\{z \in \mathbb{R}^n \mid T^{-1}z \in [x, y]\right\} \quad (2)$$

denotes the parallelotope defined by the endpoints x and y and shape matrix T. Given $a = (x, y) \in \mathbb{R}^{2n}$ with $x \preceq y$, let $[\![a]\!]$ denote the hyperrectangle formed by the first and last n components of a, $[\![a]\!] := [x, y]$, and likewise $[\![a]\!]_T := [x, y]_T$. Let $\preceq_{\mathrm{SE}}$ denote the *southeast order* on $\mathbb{R}^{2n}$ defined by

$$(x, x') \preceq_{\mathrm{SE}} (y, y') \;\Leftrightarrow\; x \preceq y \text{ and } y' \preceq x' \quad (3)$$

where $x, y, x', y' \in \mathbb{R}^n$.

III. PRELIMINARIES

We consider a dynamical system with disturbances

$$\dot{x} = F(x, w) \quad (4)$$

with state $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and disturbance input $w \in \mathcal{W} \subset \mathbb{R}^m$, where $\mathcal{W} = [\underline{w}, \overline{w}]$ for some $\underline{w} \preceq \overline{w}$.

Let $\Phi(t; x, \mathbf{w}) \in \mathcal{X}$ denote the unique state of (4) reached at time t when starting from state x at time 0 and evolving subject to the piecewise continuous signal $\mathbf{w} : [0, t] \to \mathcal{W}$. We allow for finite-time escape so that $\Phi(t; x, \mathbf{w})$ need not exist for all t, however, $\Phi(t; x, \mathbf{w})$ is understood to exist only when $\Phi(\tau; x, \mathbf{w}) \in \mathcal{X}$ for all $\tau \in [0, t]$, and statements involving $\Phi(t; x, \mathbf{w})$ are understood to apply only when $\Phi(t; x, \mathbf{w})$ exists. For given $\mathcal{X}_0 \subseteq \mathcal{X}$ and $t \geq 0$, we denote by $R(t; \mathcal{X}_0)$ the time-t forward reachable set of (4) from $\mathcal{X}_0$:

$$R(t; \mathcal{X}_0) := \{\Phi(t; x, \mathbf{w}) \in \mathcal{X} \mid x \in \mathcal{X}_0, \mathbf{w} : [0, t] \to \mathcal{W}\}. \quad (5)$$

We begin by recalling fundamental results in mixed-monotone systems theory.

Definition 1. [2] Given a locally Lipschitz continuous function $d : \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$, the system (4) is *mixed-monotone with respect to* d if

1) For all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$, $d(x, w, x, w) = F(x, w)$ holds.
2) For all $i, j \in \{1, \cdots, n\}$, with $i \neq j$, $\frac{\partial d_i}{\partial x_j}(x, w, \widehat{x}, \widehat{w}) \geq 0$ holds for all $x, \widehat{x} \in \mathcal{X}$ and all $w, \widehat{w} \in \mathcal{W}$ such the derivative exists.
3) For all $i, j \in \{1, \cdots, n\}$, $\frac{\partial d_i}{\partial \widehat{x}_j}(x, w, \widehat{x}, \widehat{w}) \leq 0$ holds for all $x, \widehat{x} \in \mathcal{X}$ and all $w, \widehat{w} \in \mathcal{W}$ such the derivative exists.
4) For all $i \in \{1, \cdots, n\}$ and all $j \in \{1, \cdots, m\}$, $\frac{\partial d_i}{\partial w_j}(x, w, \widehat{x}, \widehat{w}) \geq 0 \geq \frac{\partial d_i}{\partial \widehat{w}_j}(x, w, \widehat{x}, \widehat{w})$ holds for all $x, \widehat{x} \in \mathcal{X}$ and all $w, \widehat{w} \in \mathcal{W}$ such the derivatives exists. ■

When (4) is mixed-monotone with respect to d, d is said to be a decomposition function for (4). Given d, the system

$$\begin{bmatrix} \dot{x} \\ \dot{\widehat{x}} \end{bmatrix} = E(x, \widehat{x}) = \begin{bmatrix} d(x, \underline{w}, \widehat{x}, \overline{w}) \\ d(\widehat{x}, \overline{w}, x, \underline{w}) \end{bmatrix} \quad (6)$$

is the *embedding system relative to* d, and E is the *embedding function relative to* d. We let $\Phi^E(t; a)$ denote the unique state of (6) reached at time $t \geq 0$ when beginning from state $a \in \mathcal{X} \times \mathcal{X}$ at time 0.

We show in the following Proposition how approximations of reachable sets for (4) are efficiently computed via a single simulation of the embedding system (6).

Proposition 1. *[11] Let* (4) *be mixed-monotone with respect to* d*, and let* $\mathcal{X}_0 = [\underline{x}, \overline{x}]$ *for some* $\underline{x}, \overline{x} \in \mathcal{X}$ *with* $\underline{x} \preceq \overline{x}$*. Then* $R(t; \mathcal{X}_0) \subseteq [\![\Phi^E(t; (\underline{x}, \overline{x}))]\!]$. ■

The mixed-monotonicity property generalises the monotonicity of dynamical systems [5], for which trajectories maintain a partial order over states.

Definition 2. [5] The system (4) is a *monotone* dynamical system if

1) For all $i, j \in \{1, \cdots, n\}$, with $i \neq j$, $\frac{\partial F_i}{\partial x_j}(x, w) \geq 0$ holds for all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$.
2) For all $i \in \{1, \cdots, n\}$ and all $j \in \{1, \cdots, m\}$, $\frac{\partial F_i}{\partial w_j}(x, w) \geq 0$ holds for all $x \in \mathcal{X}$ and all $w \in \mathcal{W}$. ■

When (4) is monotone, (4) is mixed-monotone with respect to $d(x, w, \widehat{x}, \widehat{w}) = F(x, w)$, and this decomposition function yields the tightest hyperrectangle containing $R(t; \mathcal{X}_0)$ when used with Proposition 1; that is, $[\![\Phi^E(t; (\underline{x}, \overline{x}))]\!]$ contains $R(t; [\underline{x}, \overline{x}])$ and no proper hyperrectangular subset of $[\![\Phi^E(t; (\underline{x}, \overline{x}))]\!]$ contains $R(t; [\underline{x}, \overline{x}])$. When (4) is not monotone, the application of Proposition 1 is known to provide conservative estimates of reachable sets, and it is natural to wonder whether fidelity can be improved. In the following section, we discuss the three main ways that conservatism enters the approach, and in the later sections we study methods for reducing this approximation conservatism.

IV. DISCUSSION ON CONSERVATISM IN THE METHOD

Conservatism enters Proposition 1 in three main ways: (a) when a *non-tight* decomposition function is employed, (b) when the decomposition function d varies quickly from the vector field F, and (c) when the initial set $\mathcal{X}_0$ is poorly approximated by a hyperrectangle.

Generally, a mixed-monotone system will be mixed-monotone with respect to many decomposition functions, although certain decomposition functions will provide tighter approximations of reachable sets than others when used with Proposition 1. We use the term *type-(a) conservatism* to refer to the approximation error added when a poor decomposition function is employed with Proposition 1. Type-(a) conservatism was recently explored in [12] and we provide additional analysis in Section V. In particular, we recall that every mixed-monotone system induces a unique *tight decomposition function* that provides a tighter approximation of reachable sets than any other decomposition function for (4) when used with Proposition 1. This decomposition function is defined as an optimization problem and thus may not always be practically computable. To that end, we show additionally how several, perhaps non-tight, decomposition functions for (4) can be combined to form a new decomposition function for (4) that, when used with Proposition 1, provides a tighter approximation of reachable sets than is attainable by employing either initial decomposition function.

In contrast to the case when the dynamics are monotone, employing a tight decomposition function does not guarantee that no proper hyperrectangular subset of $[\![\Phi^E(t;(\underline{x},\overline{x}))]\!]$ contains $R(t;[\underline{x},\overline{x}])$. We use the term *type-(b) conservatism* to denote the approximation error that occurs in this case, and we address type-(b) conservatism in Section VII where we show that (4) may be monotone with respect to a different partial order than that considered in Definition 2. In this case, a parallelotope approximation of $R(t;\mathcal{X}_0)$ can be derived such that no proper parallelotope subset of this approximation contains $R(t;\mathcal{X}_0)$, thus mitigating type-(b) conservatism in the approach.

Lastly, conservatism can enter the method when $\mathcal{X}_0$ is poorly approximated by a hyperrectangle. While the hypothesis of Proposition 1 assumes a hyperrectangular set of interest $\mathcal{X}_0$, the basic procedure holds for different set geometries by over-approximating the initial set with a hyperrectangle; in particular, if $\mathcal{X}_0 \subset [\underline{x},\overline{x}]$ for some $\underline{x} \preceq \overline{x}$, then $R(t;\mathcal{X}_0) \subseteq [\![\Phi^E(t;(\underline{x},\overline{x}))]\!]$. However, if $[\underline{x},\overline{x}]$ poorly approximates $\mathcal{X}_0$, then $[\![\Phi^E(t;(\underline{x},\overline{x}))]\!]$ will poorly approximate $R(t;\mathcal{X}_0)$, and this approximation conservatism is referred to as *type-(c) conservatism*. In Sections VII and VIII, we show how alternative partial orders on $\mathcal{X}$, as in those discussed previously, allow for ways of reducing type-(c) conservatism.

It is important to note that reachable set approximations derived from Proposition 1 may be conservative, even when types-(a), (b) and (c) conservatism are absent. That is, even when a tight decomposition function is used, the system (4) is monotone, and $\mathcal{X}_0$ is hyperrectangular, one will generally find that $R(t;\mathcal{X}_0) \neq [\![\Phi^E(t;(\underline{x},\overline{x}))]\!]$. This approximation conservatism, referred to hereafter as *type-(d) conservatism*, is inherent in Proposition 1 and cannot be mitigated using the theory discussed thus far. We address type-(d) conservatism in Section VII; we observe, in particular, that $R(t;\mathcal{X}_0)$ is constrained to the intersection of several independent approximations derived from related systems to (6), and we show through example how forming reachable set approximations in this way mitigates type-(d) conservatism.

We summarise the proceeding discussion in Remark 1.

Remark 1. Four main forms of conservatism arise in the application of Proposition 1:

(a) Type-(a) conservatism occurs when a non-tight decomposition function is used.
(b) Type-(b) conservatism occurs when the decomposition function d varies quickly from the vector field F.
(c) Type-(c) conservatism occurs when the initial set $\mathcal{X}_0$ is poorly approximated by a hyperrectangle.
(d) Type-(d) conservatism is inherent to Proposition 1. ■

V. Reducing Conservatism via Decomposition Function Analysis

Addressing type-(a) conservatism caused by a poor choice of decomposition function for (4) requires constructing an alternative decomposition function for the same system. This issue was recently explored in [12] where it was shown that all systems of the form (4) are mixed-monotone with a unique *tight decomposition function* that provides a tighter approximation of reachable sets than any other decomposition function for (4) when used with Proposition 1; see also [13] for a discrete-time analog. Thus, applying Proposition 1 with a tight decomposition function ensures that the procedure does not suffer from type-(a) conservatism.

Despite this novelty, tight decomposition functions, as in those defined in [12], [13], are constructed as a point-wise-in-time optimization problem and, for this reason, computing alternative decomposition functions for (4) may be useful; see [4], [6], [7], [14] for generating decomposition functions from bounds on the systems Jacobian matrices.

Our first result is to show how two initial, perhaps non-tight, decomposition functions for a given system can be combined in a piecewise fashion to create a new decomposition function for the same system that approximates reachable sets with greater accuracy than either of its components.

Proposition 2. *Let* (4) *be mixed-monotone with respect to both* d^1 *and* d^2. *Then* (4) *is mixed-monotone with respect to* d *defined element-wise by*

$$d_i(x,w,\widehat{x},\widehat{w}) = \begin{cases} \max\{d_i^1(x,w,\widehat{x},\widehat{w}), d_i^2(x,w,\widehat{x},\widehat{w})\} \\ \qquad\qquad \text{if } (x,w) \preceq (\widehat{x},\widehat{w}), \\ \min\{d_i^1(x,w,\widehat{x},\widehat{w}), d_i^2(x,w,\widehat{x},\widehat{w})\} \\ \qquad\qquad \text{if } (\widehat{x},\widehat{w}) \preceq (x,w). \end{cases} \tag{7}$$

Moreover, denoting by E, E^1, E^2 *the embedding functions relative to* d, d^1, d^2, *respectively, we have that*

$$[\![\Phi^E(t;(\underline{x},\overline{x}))]\!] \subseteq [\![\Phi^{E^1}(t;(\underline{x},\overline{x}))]\!] \cap [\![\Phi^{E^2}(t;(\underline{x},\overline{x}))]\!] \tag{8}$$

for all $t \geq 0$ *and all* $\underline{x} \preceq \overline{x}$. ■

Proposition 2 shows how multiple decomposition functions for (4) are combined to construct a new decomposition function for (4); this new decomposition function, when used with Proposition 1, provides tighter approximations of reachable sets than are attainable by, in particular, employing both initial decomposition functions and forming a reachable set approximation as the intersection of the approximation derived from each function, and type-(a) conservatism in reduced using this approach. This method for reducing conservatism is particularly useful, when, say, d^1 and d^2 are both derived using the Jacobian bound approach appearing in [6], [7], [14]; applying this approach can produce multiple distinct decomposition functions for the same system and these decomposition functions can be combined using Proposition 2 to allow for added fidelity.

While employing d from (7) reduces type-(a) conservatism in the method, this approach is still susceptible to types-(b), (c) and (d) conservatism. In Section VII, we take a different approach and show how types-(b), (c) and (d) conservatism is reduced by considering multiple partial orders on $\mathcal{X}$.

VI. Applying the Tools of Mixed-Monotonicity with Alternate Partial Orders

Consider the state transformation of (4) formed by

$$y = T^{-1}x \tag{9}$$

where $x \in \mathcal{X}$ is the state of (4) and where $T \in \mathbb{R}^{n\times n}$ is a nonsingular transformation matrix. Under the transformation (9), the transformed dynamics of y become

$$\dot{y} = T^{-1}F(Ty, w) := F_T(y, w) \tag{10}$$

with state y and disturbance input $w \in \mathcal{W}$. Further, the systems (4) and (10) are related in the following way: for all $x \in \mathcal{X}$, all $t \geq 0$ and all piecewise continuous $\mathbf{w} : [0, t] \to \mathcal{W}$, we have $\Phi(t; x, \mathbf{w}) = T\Psi(t; T^{-1}x, \mathbf{w})$, where Ψ denotes the state transition function for (10).

We show next how a decomposition function for (10) enables the approximation of forward reachable sets for (4).

Theorem 1. *For some nonsingular $T \in \mathbb{R}^{n\times n}$, let* (10) *be mixed-monotone with respect to d and let $\mathcal{X}_0 = [\underline{y}, \overline{y}]_T \subseteq \mathcal{X}$ for some $\underline{y} \preceq \overline{y}$. Then $R(t; \mathcal{X}_0) \subseteq [\![\Phi^E(t; (\underline{y}, \overline{y}))]\!]_T$, where $R(t; \mathcal{X}_0)$ denotes the reachable set of the original dynamics* (4) *as defined in* (5) *and Φ^E denotes the flow of the embedding system constructed from d as defined in* (6). ■

The results of Theorem 1 subsume those of Proposition 1 as a special case by taking T to be the $n \times n$ identity matrix. We demonstrate the application of Theorem 1 in the following example.

Example 1. Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = F(x, w) = \begin{bmatrix}x_1x_2 + w\\ x_1 + 1\end{bmatrix} \tag{11}$$

with $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{W} = [0, 1/4]$. Assume a parallelotope set of initial conditions $\mathcal{X}_0 = [\underline{y}, \overline{y}]_T$ for $\underline{y} = (0, -1/4)$, $\overline{y} = (1/4, 0)$ and $T = \left[\begin{smallmatrix}1 & -2\\ 1 & 1\end{smallmatrix}\right]$, and we aim to approximate.

A tight decomposition function for (10) is formed, and its embedding system is simulated forward in time in order to approximate $R(1, \mathcal{X}_0)$. We show $\mathcal{X}_0$ graphically in Figure 1, along with $R(1; \mathcal{X}_0)$ and its respective approximation as derived in Theorem 1.

Note that the approximations derived thus far do not suffer from types-(a) and (c) conservatism; this is due to the fact that $\mathcal{X}_0$ is parallelotopic and we employ a tight decomposition function in the procedure. However, types-(b) and (d) conservatism are present and $R(1; \mathcal{X}_0) \neq [\![\Phi^E(1; [\underline{y}, \overline{y}])]\!]_T$.

The aforementioned procedure for computing parallelotope approximations of forward reachable sets can be extended to approximate backward reachable sets in a similar way [11]. An example is shown in Figure 1 where a tight decomposition function for $\dot{y} = -F_T(y, w)$ is used to over-approximate backward-time reachable sets of (11). ■

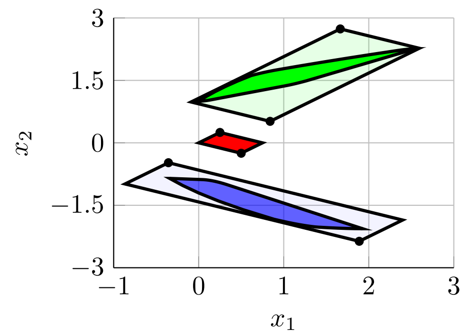


Fig. 1: Example 1: $\mathcal{X}_0$ is shown in red. $R(1; \mathcal{X}_0)$ is shown in green, with a parallelogram over-approximation shown in light green. Note that $R(1; \mathcal{X}_0) \neq [\![\Phi^E(1; [\underline{y}, \overline{y}])]\!]_T$ and this is due to types-(b) and (d) conservatism in the method. A parallelogram approximation of the time-1 backward reachable set of (11) is also computed using a tight decomposition function for $\dot{y} = -F_T(y, w)$ as described in [11] and is shown in blue.

Theorem 1 induces an analogous notion of conservatism to that of Proposition 1. That is, (a) one may not have access to a tight decomposition function for (10), (b) T may be chosen poorly so that the system (10) may only induce decomposition functions which vary quickly from F_y, and (c) the set of interest $\mathcal{X}_0$ may be poorly approximated by a parallelotope $[\underline{y}, \overline{y}]_T$.

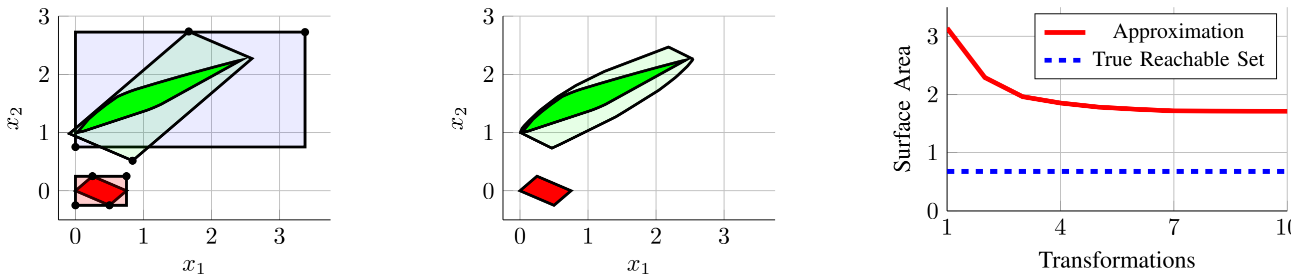
VII. Reducing Conservatism via the Use of Multiple Partial Orders

The main focus of this paper is to discover means of improving fidelity in the approximations derived from the application of Proposition 1, and we have shown previously how type-(a) conservatism is assessed using decomposition function analysis. We next show how multiple partial orders, as in those discussed in Section VI, can be employed to reduce types-(b), (c) and (d) conservatism.

We first turn our attention to type-(d) conservatism. As suggested in the previous discussion, a naive approach for deriving tighter approximations of reachable sets is to construct several decomposition functions for (4) and then form an approximation of the reachable set of (4) as the intersection of the approximations derived from each decomposition function. This unnecessarily complicates the approach since, by Proposition 2, multiple decomposition functions for (4) can be combined to form a decomposition function that achieves approximations of reachable sets at least as tight. Moreover, the application of (7) is still subject to type-(d) conservatism, as this approximation conservatism is inherent.

Nonetheless, we show in Example 2 how type-(d) conservatism is mitigated when decompositions for (4) and (10) are used together to approximate reachable sets.

Example 2. We consider the system (11), previously studied in Example 1. We take $T = \left[\begin{smallmatrix}1 & -2\\ 1 & 1\end{smallmatrix}\right]$ and $\mathcal{X}_0 = [\underline{y}, \overline{y}]_T$ and we aim to approximate $R(1, \mathcal{X}_0)$ by applying Theorem 1.



(a) (b) (c)

Fig. 2: Example 2: Approximating $R(1;\mathcal{X}_0)$ by applying Proposition 1 and Theorem 1. (a) Comparing Proposition 1 to Theorem 1: $\mathcal{X}_0$ is shown in red. $R(1;\mathcal{X}_0)$ is shown in green, with a parallelogram over-approximation shown in light green. $\mathcal{X}_1$ is shown in pink, and a rectangular over-approximation of $R(1;\mathcal{X}_1)$ shown in blue. (b) Increasing fidelity with multiple transformations: $\mathcal{X}_0$ is shown in red. $R(1;\mathcal{X}_0)$ is shown in green. An approximation of $R(1;\mathcal{X}_0)$ is formed by computing the intersection of 10 approximations derived via Theorem 1. This approximation is shown in light green. (c) The red line depicts the surface area of the reachable set approximation as a function of the number of transformation used. After all 10 applications of Theorem 1, the surface area of the resulting approximation is 1.71. The blue line depicts the surface area of the true reachable set $R(1;\mathcal{X}_0)$, which is equal to 0.67.

Here, $\mathcal{X}_0$ is parallelotopic and, therefore, types-(a) and (c) conservatism are absent in the application of Theorem 1. Further, it is possible to reduce type-(d) conservatism by applying Theorem 1 several times with different transformations, so that $R(1,\mathcal{X}_0)$ is constrained to the intersection of each approximation derived. As a demonstration, we take $\mathcal{X}_1 = [0,3/4]\times[-1/4,1/4]$, so that $\mathcal{X}_0 \subset \mathcal{X}_1$, and compute a rectangular over-approximation of $R(1;\mathcal{X}_0)$ by applying Proposition 1 with a decomposition function for (11). This study is depicted graphically in Figure 2a. Note that applying Proposition 1 is subject to type-(c) conservatism as $\mathcal{X}_0$ is not rectangular and the initial application of Theorem 1 produced a significantly tighter approximation of $R(1;\mathcal{X}_0)$. Nonetheless, fidelity is best improved when Proposition 1 and Theorem 1 are employed together.

To illustrate this point further, we next form an approximation of $R(1;\mathcal{X}_0)$ by applying Theorem 1 with 10 different transformations matrices; an approximation of $R(1;\mathcal{X}_0)$ is then formed as the intersection of the approximation derived from each transformation (See Figures 2b–2c). ■

It is important to note also that, in certain instances, a transformation T can be chosen so that (10) is a monotone system and, in this instance, the application of Theorem 1 is devoid of type-(b) conservatism. In this case, a parallelotope approximation of $R(t;\mathcal{X}_0)$ can be derived such that no proper parallelotope subset of this approximation contains $R(t;\mathcal{X}_0)$.

Example 3. Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = F(x,w) = \begin{bmatrix} x_1 - x_2 + x_2^3 + w \\ x_1 - x_2\end{bmatrix} \qquad (12)$$

with $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{W} = [-1,1]$. Under the transformation $T_1 = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ the dynamics of y from (10) become

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = F_{T_1}(y,w) = \begin{bmatrix} y_2^3 + w \\ y_1\end{bmatrix}, \qquad (13)$$

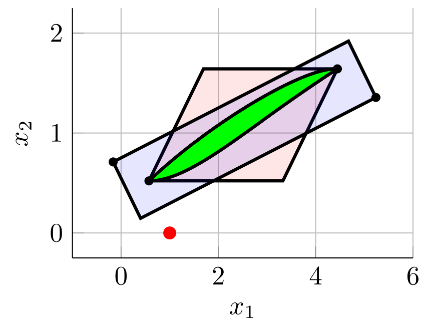


Fig. 3: Example 3: x_0 is shown in red and $R(1;x_0)$ is shown in green. Two parallelogram approximation of $R(1;x_0)$ are computed using Theorem 1 with T_1 and T_2 and these approximations are shown in pink and blue, respectively. Note that while T_1 induces a monotone system, and achieves the tightest parallelogram containing $R(1;x_0)$, conservatism is still reduced when repeating the procedure with T_2.

and (13) is a monotone system. Thus, the application of Theorem 1 with T_1 will not be subject to type-(b) conservatism. An example is shown in Figure 3, where we approximate $R(1;x_0)$ using Theorem 1, where $x_0 = (1,1)$ and a tight decomposition function for (10) with T_1 is used. Note that the approximation is not subject to types-(a), (b) and (c) conservatism as a tight decomposition function is employed, x_0 is trivially a parallelotope, and (10) is monotone. Nonetheless, the approximation still contains type-(d) conservatism.

Even though (12) is transformable to a monotone system via T_1, fidelity in the approximation can still be improved by applying Theorem 1 again with a different shape matrix. An example is shown in Figure 3, where we compare the approximation derived with T_1 to a second approximation derived using $T_2 = \left[\begin{smallmatrix}1 & 4\\ -1 & 1\end{smallmatrix}\right]$. ■

Examples 2 and 3 demonstrate a new approach for reducing conservatism in the application of the mixed-monotonicity property. This approach involves computing a polytope approximation of $R(t;\mathcal{X}_0)$ as the intersection of several parallelotope approximations derived via Theorem 1. In the next section, we present a numerical example and demonstrate a novel method for reducing type-(c) conservatism when the initial set $\mathcal{X}_0$ is polytopic.

VIII. Numerical Example

Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = F(x,w) = \begin{bmatrix}x_2+\sin(x_2)+w\\ x_1+\cos(x_1)+1\end{bmatrix} \tag{14}$$

with $\mathcal{X}=\mathbb{R}^2$ and $\mathcal{W}=[0,1/2]$, and consider a hexagonal set of initial conditions

$$\mathcal{X}_0 = \textbf{Conv}\{x\in\mathbb{R}^2 \mid x_1 = 1+\cos(\tfrac{i\pi}{3}),\; x_2 = 1+\sin(\tfrac{i\pi}{3}), i\in\{1,\cdots,6\}\} \tag{15}$$

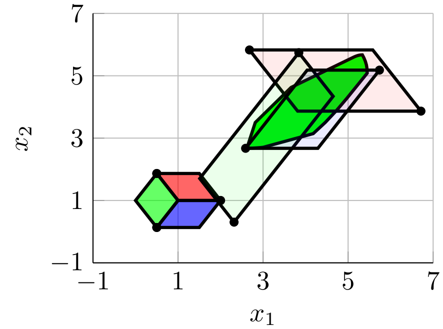
where **Conv** denotes the convex hull function. We aim to overapproximate $R(1;\mathcal{X}_0)$.

Note that $\mathcal{X}_0$ can be described as the union of three *disjoint* parallelograms, as shown in Figure 4a. For each of the three parallelogram geometries, a tight decomposition function is formed for (10) and the time-1 reachable set of (10) is approximated using Theorem 1. An approximation $R(1;\mathcal{X}_0)$ is then formed as the union of the three approximations derived here. This procedure is also repeated using three *overlapping* parallelograms, as shown in Figure 4b.

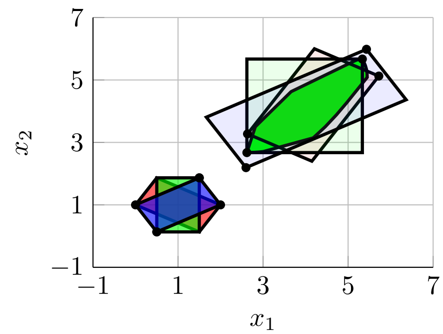
In summary, the previous example demonstrates how polygon initial sets that accommodated similarly by Theorem 1, and this procedure avoids type-(c) conservatism in the approach. This procedure is applicable to all systems (4) and all polytope initial sets $\mathcal{X}_0$ with hyperrectangular faces.

IX. Conclusion

This work studies means of improving fidelity in the approximations of reachable sets for nonlinear systems using the theory of mixed-monotonicity. Four main forms of conservatism are considered, and we show how applying the tools of mixed-monotonicity to a related system, formed via a linear transformation of the initial state-space, is used to reduce this conservatism.



(a) The initial set $\mathcal{X}_0$ is the union of three disjoint parallelograms $\mathcal{X}_0^1$, $\mathcal{X}_0^2$ and $\mathcal{X}_0^3$, shown red, green and blue. The reachable set of each is approximated using theorem 1 and is shown in respective colors. The true reachable set $R(1;\mathcal{X}_0)$ is shown in green.



(b) The initial set $\mathcal{X}_0$ is the union of three overlapping parallelograms, shown red, green and blue. The reachable set of each is approximated using theorem 1 and is shown in respective colors. The true reachable set $R(1;\mathcal{X}_0)$ is shown in green.

Fig. 4: Numerical Example: Approximating $R(1;\mathcal{X}_0)$ where $\mathcal{X}_0$ is the union of parallelograms.

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