Computing Robustly Forward Invariant Sets for Mixed-Monotone Systems

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***Abstract*— Safety for dynamical systems is often posed as an invariance constraint, requiring the system trajectory to remain in some safe subset of the state-space for all time. This note presents new tools for studying reachability and set invariance for nondeterministic systems subject to a disturbance input using the theory of mixed-monotone dynamical systems. The vector field of a mixed-monotone system is characterized as being decomposable into increasing and decreasing components which allows the dynamics to be embedded in a higher dimensional embedding system. Even though the original system is nondeterministic due to the unknown disturbance input, the embedding system has no disturbance and a single simulation of the embedding system provides bounds for reachable sets of the original dynamics.**

In this paper, we present an efficient method for identifying robustly forward invariant and attractive sets for mixed-monotone systems by studying equilibria and their stability properties of the corresponding embedding system. We show how this approach can be applied to either the backward-time dynamics or a set of linearly transformed dynamics to establish different robustly forward invariant sets for the original dynamics, and we show also how periodic solutions to the embedding system establish invariant regions for the original dynamics as well. The findings of this work are demonstrated through two numerical examples and two case studies, including a five-dimensional planar quadrotor system.

***Index Terms*— Mixed-monotone systems, Stability of nonlinear systems, Computational methods, Uncertain systems**

I. Introduction

SAFETY for dynamical systems is often posed as an invariance constraint requiring the system trajectory to remain in some safe subset of the state-space for all time. Given a candidate subset, forward invariance can be shown by, *e.g.*, studying the vector field on the boundary of the set [1] or using barrier certificates [2]; however, it is generally difficult to identify such candidates without explicit domain knowledge of the system in question. In this work, we provide several tools for identifying robustly forward invariant and attractive sets for *mixed-monotone* systems subject to a disturbance input.

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A dynamical system is mixed-monotone if there exists a related *decomposition function* that decomposes the system's vector field into increasing and decreasing components; mixed-monotonicity applies to continuous-time systems [3]–[6], discrete-time systems [7], as well as systems with disturbances [8]–[10], and it generalizes the *monotonicity* property of dynamical systems for which trajectories maintain a partial order over states [11], [12].

In the case with no disturbance, it is well known that a $2n$-dimensional symmetric *embedding system* can be constructed from the decomposition function of an n-dimensional mixed-monotone system. This embedding system is monotone with respect to a particular southeast cone and the original dynamics are contained in an invariant n-dimensional *diagonal* subspace. Thus, tools from monotone systems theory can be applied to the embedding system to conclude properties of the original dynamics; in particular, such approaches are useful for stability analysis [13], [14], reachability analysis [7], and formal verification and synthesis [15], [16].

When disturbances are present, it is also possible to construct a monotone embedding system from the original dynamics. Here, the embedding system is nondeterministic with a $2m$-dimensional disturbance input when the original system is subject to an m-dimensional disturbance input. This result has been applied in discrete-time [9], [10], and in continuous-time [8], [9]; in these works, reachable sets are computed from trajectories of the embedding system using fundamental results for monotone control systems as described in [12].

In this technical note, we consider continuous-time mixed-monotone systems with disturbances, however, unlike [8]–[10] we study a *deterministic* embedding system that arises from considering the worst case disturbance inputs. While this deterministic embedding system is straightforwardly derived from the aforementioned nondeterministic embedding system, its potential does not seem to have been fully appreciated or studied in the literature. In particular, unlike the deterministic embedding system that arises in the case with no disturbance, the diagonal of this new deterministic embedding system is not forward invariant; instead, a forward invariant *triangular* region is induced above the diagonal. An important observation made in this work is that equilibria in this triangular region correspond to robustly forward invariant sets for the original system. Additionally, these equilibria can be globally asymptotically stable—which is not possible in the case with no disturbance—and we show that stable equilibria correspond to attractive sets for the original system. We apply this observation and construct an efficient algorithm for identifying robustly forward invariant sets and attractive sets for mixed-monotone systems and this procedure is well suited for safety applications requiring such knowledge: see [17], [18] for examples of safety applications for mixed-monotone systems. We show also how this approach can be applied to a linearly transformed set of dynamics to identify parallelotope forward invariant regions for the original system and can be applied to the backward-time dynamics to identify forward invariant regions that are the complement of hyperrectangles.

This work extends the paper [19], which shows how hyperrectangular robustly forward invariant sets for mixed-monotone systems are efficiently identified via the computation of an equilibrium for the related deterministic embedding system. In the present technical note, we extend this result by including a necessary condition for the existence of robustly forward invariant sets for the original system, and we show through example how parallelotopic robustly forward invariant sets for the original dynamics are obtained by applying the results of [20]. We demonstrate these results with a case study in which forward invariant parallelotope regions are computed for a 5-dimensional planar quadrotor system[1]. As a second extension, we show via example how the embedding system of a nondeterministic mixed-monotone system can induce a periodic orbit above the diagonal, which is also not possible in the case without disturbances. In this case, a robustly forward invariant region for the initial mixed-monotone system can be constructed from the periodic orbit, and stability of the orbit implies attractiveness of the resulting invariant set.

II. Notation

Let (x, y) denote the vector concatenation of $x, y \in \mathbb{R}^n$, *i.e.*, $(x, y) := [x^T\, y^T]^T \in \mathbb{R}^{2n}$, and let $\preceq$ denote the componentwise vector order, *i.e.*, $x \preceq y$ if and only if $x_i \le y_i$ for all $i \in \{1, \cdots, n\}$ where vector components are indexed via subscript. Given $x, y \in \mathbb{R}^n$ with $x \preceq y$,

$$[x, y] := \{z \in \mathbb{R}^n \mid x \preceq z \text{ and } z \preceq y\}$$

denotes the hyperrectangle defined by the endpoints x and y. We also allow $x_i \in \mathbb{R} \cup \{-\infty\}$ and $y_i \in \mathbb{R} \cup \{\infty\}$, in which case $[x, y]$ defines an *extended hyperrectangle*, that is, a hyperrectangle with possibly infinite extent in some coordinates. Given $a = (x, y) \in \mathbb{R}^{2n}$ with $x \preceq y$, we denote by $[\![a]\!]$ the hyperrectangle formed by the first and last n components of a, *i.e.*, $[\![a]\!] := [x, y]$.

Let $\preceq_{\mathrm{SE}}$ denote the *southeast order* on $\mathbb{R}^{2n}$ defined by

$$(x, x') \preceq_{\mathrm{SE}} (y, y') \;\Leftrightarrow\; x \preceq y \text{ and } y' \preceq x'$$

where $x, y, x', y' \in \mathbb{R}^n$. In the case that $x \preceq x'$ and $y \preceq y'$, observe that

$$(x, x') \preceq_{\mathrm{SE}} (y, y') \;\Leftrightarrow\; [y, y'] \subseteq [x, x']. \tag{1}$$

[1]The code that accompanies the examples in this work, and generates the figures, is publicly available through the GaTech FactsLab GitHub: `https://github.com/gtfactslab/Abate_TAC2021`.

III. Mathematical Preliminaries

In this section, we present the definition of mixed-monotonicity in continuous-time, and we show how reachable sets for mixed-monotone systems are efficiently over-approximated via a single simulation of a related monotone embedding system constructed from the initial dynamics.

A. Problem Setting

We consider a dynamical system with disturbance input, *i.e.*, a nondeterministic system, given by

$$\dot{x} = F(x, w) \tag{2}$$

for Lipschitz F where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $w \in \mathcal{W} \subset \mathbb{R}^m$ denote the system state and a bounded time-varying disturbance, respectively. We assume $\mathcal{X}$ is an extended hyperrectangle with nonempty interior and $\mathcal{W}$ is a hyperrectangle[2] so that $\mathcal{W} = [\underline{w}, \overline{w}]$ for some $\underline{w}, \overline{w} \in \mathbb{R}^m$ with $\underline{w} \preceq \overline{w}$.

For $t \ge 0$, let $\Phi^F(t; x, \mathbf{w})$ denote the state of (2) reached at time t starting from $x \in \mathcal{X}$ at time 0 under the piecewise continuous disturbance input $\mathbf{w} : [0, t] \to \mathcal{W}$; throughout, we always assume that time-varying disturbance signals such as $\mathbf{w}$ are piecewise continuous so that, in particular, $\Phi^F(t; x, \mathbf{w})$ is unique, although generally we do not explicitly state this restriction. We do not a priori require $\Phi^F(t; x, \mathbf{w})$ to exist for all t; however, existence of $\Phi^F(t; x, \mathbf{w})$ implicitly means that $\Phi^F(t; x, \mathbf{w}) \in \mathcal{X}$ for all $0 \le \tau \le t$. Additionally, let

$$R^F(t; \mathcal{X}_0) := \Big\{ \Phi^F(t; x, \mathbf{w}) \in \mathcal{X} \;\Big|\; x \in \mathcal{X}_0 \text{ for some } \mathbf{w} : [0, t] \to \mathcal{W} \Big\} \tag{3}$$

denote the set of states that are reachable by (2) in time $t \ge 0$ from $\mathcal{X}_0 \subseteq \mathcal{X}$ under some disturbance input.

A key focus of this paper is in devising efficient algorithms for the computation of *robustly forward invariant sets* and *attractive sets* for (2). Definition 2 is taken from [21].

Definition 1. A set $A \subseteq \mathcal{X}$ is *robustly forward invariant* for (2) if $\Phi^F(t; x, \mathbf{w}) \in A$ for all $x \in A$, all $t \ge 0$ and all piecewise continuous disturbance inputs $\mathbf{w} : [0, t] \to \mathcal{W}$ whenever $\Phi^F(t; x, \mathbf{w})$ exists. When F does not depend on w we simply say A is *forward invariant*. ■

Definition 2. A set $A \subset \mathcal{X}$ is *attractive from* $\mathcal{X}' \subset \mathcal{X}$, or simply *attractive* for (2), if for each solution $\Phi^F(\,\cdot\,; x, \mathbf{w})$ to (2) with $x \in \mathcal{X}'$ and piecewise continuous $\mathbf{w}$ and each relatively open neighborhood $\mathcal{X}_\epsilon \subset \mathcal{X}$ of A, there exists $\tau > 0$ such that $\Phi^F(t; x, \mathbf{w}) \in \mathcal{X}_\epsilon$ for all $t \ge \tau$. When $\mathcal{X}' = \mathcal{X}$, we say A is *globally attractive*. ■

B. Mixed-Monotone Systems

We next present the definition of mixed-monotonicity in continuous-time.

Definition 3. [22] Given a locally Lipschitz continuous function $d : \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$, the system (2) is

[2]The assumption that $\mathcal{X}$ is an extended hyperrectangle and $\mathcal{W}$ is a hyperrectangle can be relaxed for some of the results of this paper, but for ease of exposition, we make this assumption throughout.

mixed-monotone with respect to d if for all $x, \widehat{x} \in \mathcal{X}$ and all $w, \widehat{w} \in \mathcal{W}$ the following hold:

- $d(x, w, x, w) = F(x, w)$,
- $\frac{\partial d_i}{\partial x_j}(x, w, \widehat{x}, \widehat{w}) \geq 0$ for all $i, j \in \{1, \cdots, n\}$ with $i \neq j$,
- $\frac{\partial d_i}{\partial \widehat{x}_j}(x, w, \widehat{x}, \widehat{w}) \leq 0$ for all $i, j \in \{1, \cdots, n\}$,
- $\frac{\partial d_i}{\partial w_k}(x, w, \widehat{x}, \widehat{w}) \geq 0$ for all $i \in \{1, \cdots, n\}$ and all $k \in \{1, \cdots, m\}$,
- $\frac{\partial d_i}{\partial \widehat{w}_k}(x, w, \widehat{x}, \widehat{w}) \leq 0$ for all $i \in \{1, \cdots, n\}$ and all $k \in \{1, \cdots, m\}$. ■

If (2) is mixed-monotone with respect to d, d is said to be a *decomposition function* for (2), and when d is clear from context we simply say (2) is mixed-monotone. In the special instance where the system (2) is mixed-monotone with respect to d given by $d(x, w, \widehat{x}, \widehat{w}) = F(x, w)$, Definition 3 recovers familiar conditions establishing *monotonicity* [11], [12]. A common interpretation of monotonicity is requiring cooperative interaction among all state variables, and mixed-monotonicity provides an extension allowing for competitive effects captured by the hatted variables.

Remark 1. While decomposition functions are generally defined on $\mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W}$, it is observed in [23] that the common reachability analysis tools for mixed-monotone systems require d to be evaluable only when its inputs are ordered, *i.e.*, when $(x, w) \preceq (\widehat{x}, \widehat{w})$ or $(\widehat{x}, \widehat{w}) \preceq (x, w)$. In this work, we sometimes construct decomposition functions only on the space of ordered inputs using, for example, the tight decomposition function construction [23], which appears later in Proposition 1. ■

C. On the Construction of Decomposition Functions

The question of existence of decomposition functions was recently studied in [23] where it was established that all dynamical systems (2), possessing a Lipschitz-continuous vector field, are mixed-monotone with a decomposition function constructed in the following way.

Proposition 1. *[23] Any system of the form* (2) *is mixed-monotone with respect to* d *constructed elementwise according to*

$$d_i(x, w, \widehat{x}, \widehat{w}) = \begin{cases} \displaystyle\min_{\substack{y \in [x, \widehat{x}] \\ y_i = x_i \\ z \in [w, \widehat{w}]}} F_i(y, z) & \text{if } (x, w) \preceq (\widehat{x}, \widehat{w}), \\ \displaystyle\max_{\substack{y \in [\widehat{x}, x] \\ y_i = x_i \\ z \in [\widehat{w}, w]}} F_i(y, z) & \text{if } (\widehat{x}, \widehat{w}) \preceq (x, w). \end{cases} \tag{4}$$

We refer to the unique decomposition function formed in (4) as the *tight decomposition function* for (2). As posed in (4), computing a tight decomposition function requires solving a generally nonconvex optimization problem for each quadruple $(x, w, \widehat{x}, \widehat{w})$. However, in certain instances it is possible to compute a tight decomposition function in closed form, and we demonstrate this assertion later through example. Regardless, the general computational infeasibility of (4) implies that it is of limited direct use and, in practice, decomposition functions are generally not obtained using Proposition 1. See [5], [9], [24] for an algorithm to construct decomposition functions for systems with uniformly bounded Jacobian matrices, and see also [19] for an algorithm to construct decomposition functions for systems with polynomial vector fields.

D. Efficient Reachability Analysis via the Mixed-Monotonicity Property

The key feature of mixed-monotone systems that we exploit in this paper is that over-approximations of reachable sets can be efficiently computed by considering a deterministic auxiliary system constructed from the decomposition function. We first consider the nondeterministic system

$$\begin{bmatrix} \dot{x} \\ \dot{\widehat{x}} \end{bmatrix} = \varepsilon(x, w, \widehat{x}, \widehat{w}) := \begin{bmatrix} d(x, w, \widehat{x}, \widehat{w}) \\ d(\widehat{x}, \widehat{w}, x, w) \end{bmatrix} \tag{5}$$

with state $(x, \widehat{x}) \in \mathcal{X} \times \mathcal{X}$ and disturbance input $(w, \widehat{w}) \in \mathcal{W} \times \mathcal{W}$. We call (5) the *embedding system* relative to d, and we denote by $\Phi^{\varepsilon}(t; (\underline{x}, \overline{x}), (\mathbf{w}, \widehat{\mathbf{w}}))$ the state of (5) at time t when initialized at $(\underline{x}, \overline{x}) \in \mathcal{X} \times \mathcal{X}$ and when subjected to the piecewise continuous inputs $\mathbf{w}, \widehat{\mathbf{w}} : [0, t] \to \mathcal{W}$.

There are two important structural features of the nondeterministic embedding system (5). First, (5) is a monotone control system as defined in [12] when the orders on $\mathcal{X} \times \mathcal{X}$ and $\mathcal{W} \times \mathcal{W}$ are both taken to be the southeast orders; that is, if $a, a' \in \mathcal{X} \times \mathcal{X}$ and $\mathbf{b}, \mathbf{b}' : [0, \infty) \to \mathcal{W} \times \mathcal{W}$ satisfy $a \preceq_{\mathrm{SE}} a'$ and $\mathbf{b}(t) \preceq_{\mathrm{SE}} \mathbf{b}'(t)$ for all $t \geq 0$, then

$$\Phi^{\varepsilon}(t; a, \mathbf{b}) \preceq_{\mathrm{SE}} \Phi^{\varepsilon}(t; a', \mathbf{b}') \tag{6}$$

for all $t \geq 0$, provided $\Phi^{\varepsilon}(\,\cdot\,; a, \mathbf{b})$ and $\Phi^{\varepsilon}(\,\cdot\,; a', \mathbf{b}')$ remain in $\mathcal{X} \times \mathcal{X}$ on $[0, t]$ [12]. Second, the nondeterministic embedding system (5) is *symmetric* in the sense that

$$\Phi^{\varepsilon}(t; (x, \widehat{x}), (\mathbf{w}, \widehat{\mathbf{w}})) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \Phi^{\varepsilon}(t; (\widehat{x}, x), (\widehat{\mathbf{w}}, \mathbf{w})) \tag{7}$$

holds for all $t \geq 0$, all $x \preceq \widehat{x}$, and all $\mathbf{w}, \widehat{\mathbf{w}} : [0, \infty) \to \mathcal{W}$ with $\mathbf{w}(t) \preceq \widehat{\mathbf{w}}(t)$ for all $t \in [0, \infty)$. An implication of symmetry is that the nondeterministic embedding system (5) induces an invariant *diagonal* space when the restriction $w = \widehat{w}$ is imposed; that is, defining

$$\Delta := \{(x, \widehat{x}) \in \mathcal{X} \times \mathcal{X} \mid x = \widehat{x}\} \tag{8}$$

as the *diagonal* of the embedding system, we have that $\Phi^{\varepsilon}(t; a, (\mathbf{w}, \mathbf{w})) \in \Delta$ for all $a \in \Delta$ and all $\mathbf{w} : [0, \infty) \to \mathcal{W}$.

Throughout most of this paper, we instead utilize a *deterministic embedding system* given by

$$\begin{bmatrix} \dot{x} \\ \dot{\widehat{x}} \end{bmatrix} = e(x, \widehat{x}) := \begin{bmatrix} d(x, \underline{w}, \widehat{x}, \overline{w}) \\ d(\widehat{x}, \overline{w}, x, \underline{w}) \end{bmatrix}. \tag{9}$$

We refer to e as the *embedding function relative to* d and we denote by $\Phi^{e}(t; a) = \Phi^{\varepsilon}(t; a, (\underline{w}, \overline{w}))$ the state transition function of (9). Note that

$$\Phi^{e}(t; a) \preceq_{\mathrm{SE}} \Phi^{e}(t; a') \tag{10}$$

for all $a, a' \in \mathcal{X} \times \mathcal{X}$ with $a \preceq_{\mathrm{SE}} a'$ and for all $t \geq 0$, *i.e.*, (9) is monotone with respect to the southeast order. However, unlike (5), the deterministic embedding system (9) is not symmetric and Δ does not generally enjoy a forward invariance property on (9) unless $\underline{w} = \overline{w}$.

We next recall the following result establishing that the reachable set R^F is over-approximated by solutions to the deterministic embedding system (9).

Proposition 2 ([9, Prop. 6]). *Let* (2) *be mixed-monotone with respect to* d, *and consider* $\mathcal{X}_0 = [\underline{x}, \overline{x}]$ *for some* $\underline{x} \preceq \overline{x}$. *If* $\Phi^e(\tau; (\underline{x}, \overline{x})) \in \mathcal{X} \times \mathcal{X}$ *for all* $0 \le \tau \le t$, *then*

$$R^F(t; \mathcal{X}_0) \subseteq [\![\Phi^e(t; (\underline{x}, \overline{x}))]\!]. \tag{11}$$

Proof. Choose $x \in [\underline{x}, \overline{x}]$ and $\mathbf{w} : [0, t] \to \mathcal{W}$ for some $t \ge 0$. Then from (5) we have

$$(\Phi^F(t; x, \mathbf{w}), \Phi^F(t; x, \mathbf{w})) = \Phi^\varepsilon(t; (x, x), (\mathbf{w}, \mathbf{w})),$$
$$\Phi^e(t; (\underline{x}, \overline{x})) = \Phi^\varepsilon(t; (\underline{x}, \overline{x}), (\underline{w}, \overline{w})).$$

Since $(\underline{x}, \overline{x}) \preceq_{\mathrm{SE}} (x, x)$ and $(\underline{w}, \overline{w}) \preceq_{\mathrm{SE}} (\mathbf{w}(\tau), \mathbf{w}(\tau))$ for all $\tau \in [0, t]$, we now have

$$\Phi^e(t; (\underline{x}, \overline{x})) \preceq_{\mathrm{SE}} \begin{bmatrix} \Phi^F(t; x, \mathbf{w}) \\ \Phi^F(t; x, \mathbf{w}) \end{bmatrix} \tag{12}$$

and thus $\Phi^F(t; x, \mathbf{w}) \in [\![\Phi^e(t; (\underline{x}, \overline{x}))]\!]$. Therefore $R^F(t; \mathcal{X}_0) \subseteq [\![\Phi^e(t; (\underline{x}, \overline{x}))]\!]$. □

Remark 2. The work [19] shows how a decomposition function for the backward-time dynamics

$$\dot{x} = -F(x, w) \tag{13}$$

enables the over-approximation of backward-time reachable sets for (2) using a similar approach. That is, letting

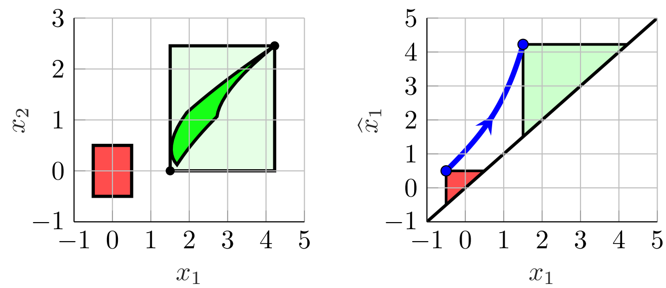
$$S^F(t; \mathcal{X}_1) := \Big\{ x \in \mathcal{X} \,\Big|\, \Phi^F(t; x, \mathbf{w}) \in \mathcal{X}_1 \text{ for some } \mathbf{w} : [0, t] \to \mathcal{W} \Big\} \tag{14}$$

denote the set of initial conditions for which there exists a $\mathbf{w} : [0, t] \to \mathcal{W}$ capable of driving (2) to the set $\mathcal{X}_1$ in time $t \ge 0$, it holds that

$$S^F(t; \mathcal{X}_1) \subseteq [\![\Phi^E(t; (\underline{x}, \overline{x}))]\!] \tag{15}$$

when $\Phi^E(\tau; (\underline{x}, \overline{x})) \in \mathcal{X} \times \mathcal{X}$ for all $0 \le \tau \le t$ and where E is the embedding function (9) constructed from a decomposition function D for the backward-time dynamics (13). We make use of this fact—and the results of Proposition 2—for the computation of robustly forward invariant regions for (2) later in Section IV. ■

It is important to note that the usefulness of the mixed-monotonicity property for stability and reachability analysis—the main focus of this paper—is dependent on the *choice* of d. In general, a mixed-monotone system will be mixed-monotone with respect to many decomposition functions; however, certain decomposition functions may be more or less conservative than others when used with Proposition 2. It is shown in [23], for instance, that the tight decomposition function (4), when used with Proposition 2, will provide the *tightest* possible approximation of R^F of any decomposition function for (2). Conservatism in the application of Proposition 2 is also a main focus of [20], [24].



(a) (b)

Fig. 1: Example 1: approximating forward reachable sets of (16) from the set of initial conditions $\mathcal{X}_0 = [-1/2, 1/2]^2$. Subfigures: (a) $\mathcal{X}_0$ is shown in red. $R^F(1; \mathcal{X}_0)$ is shown in green. The hyperrectangular over-approximation of $R^F(1; \mathcal{X}_0)$, which is computed from the embedding system (9) as described in Proposition 2, is shown in light green. (b) Visualization of the bounding procedure from Proposition 2. The trajectory of (9) that yields Figure 1a is shown in blue, where Φ^e is projected to the $x_1, \widehat{x}_1$ plane. The southeast cones corresponding to $\mathcal{X}_0$ and the hyperrectangular over-approximation of $R^F(1; \mathcal{X}_0)$ are shown in red and green, respectively.

Example 1. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = F(x) = \begin{bmatrix} x_2^2 + 2 \\ x_1 \end{bmatrix} \tag{16}$$

with $\mathcal{X} = \mathbb{R}^2$. The system (16) is mixed-monotone on $\mathcal{X}$ and a tight decomposition function is given in closed form by

$$d_1(x, \widehat{x}) = \begin{cases} x_2^2 + 2 & \text{if } x_2 \ge \max\{0, -\widehat{x}_2\}, \\ \widehat{x}_2^2 + 2 & \text{if } \widehat{x}_2 \le \min\{0, -x_2\}, \\ 2 & \text{if } x_2 \le 0 \le \widehat{x}_2, \end{cases} \tag{17}$$
$$d_2(x, \widehat{x}) = x_1.$$

Consider now a hyperrectangular set of initial conditions $\mathcal{X}_0$. Proposition 2 implies that the reachable set (3) from $\mathcal{X}_0$ is approximated by a rectangular set defined from the state transition function of the $2n$ dimensional embedding system (9). An example is shown in Figures 1a and 1b. ■

IV. Obtaining Forward Invariant and Attractive Sets for Mixed-Monotone Systems

In this section, we present our main result and show how robustly forward invariant sets and attractive sets for (2) are efficiently computed via the identification of equilibria for the deterministic embedding system (9).

A. On Deterministic Embedding Systems

While the nondeterministic embedding system (5) has appeared in the literature before, along with connections to reachable set computations, the deterministic embedding system (9) has not been fully considered or studied. We begin with two lemmas on forward invariant regions for the deterministic embedding system (9).

As stated previously, the diagonal space Δ defined in (8) is generally not forward invariant for (9). Nonetheless, we next establish that the set of states in $\mathcal{X}\times\mathcal{X}$ that lie *above* Δ with respect to the southeast order is forward invariant. Define

$$\mathcal{T} := \{(x,\widehat{x})\in\mathcal{X}\times\mathcal{X}\,|\,x\preceq\widehat{x}\} \tag{18}$$

the *upper triangle* of the embedding system.

Lemma 1. $\mathcal{T}$ is forward invariant for (9). ■

Proof. Choose $(\underline{x},\overline{x})\in\mathcal{T}$, then $\underline{x}\preceq\overline{x}$. Define

$$(x(t),\widehat{x}(t)) = \Phi^e(t;(\underline{x},\overline{x})), \tag{19}$$

where from (5) we now have

$$(x(t),\widehat{x}(t)) = \Phi^\varepsilon(t;(\underline{x},\overline{x}),(\underline{w},\overline{w})),$$
$$(\widehat{x}(t),x(t)) = \Phi^\varepsilon(t;(\overline{x},\underline{x}),(\overline{w},\underline{w})).$$

Since $(\underline{x},\overline{x})\preceq_{\mathrm{SE}}(\overline{x},\underline{x})$ and $(\underline{w},\overline{w})\preceq_{\mathrm{SE}}(\overline{w},\underline{w})$ we have $(x(t),\widehat{x}(t))\preceq_{\mathrm{SE}}(\widehat{x}(t),x(t))$ for all $t\geq 0$. Equivalently, $x(t)\preceq\widehat{x}(t)$ for all $t\geq 0$, and thus $\Phi^e(t;(\underline{x},\overline{x}))\in\mathcal{T}$. Therefore, $\mathcal{T}$ is forward invariant for (9). □

Now define

$$\mathcal{S} := \{(x,\widehat{x})\in\mathcal{T}\,|\,0\preceq_{\mathrm{SE}} e(x,\widehat{x})\} \tag{20}$$

the set of states in $\mathcal{T}$ such that the embedding system's vector field points into the southeast cone. The following lemma is a direct result of [11, Chapter 3, Proposition 2.1].

Lemma 2. The set $\mathcal{S}$ is forward invariant for (9), and $\Phi^e(t_1;a)\preceq_{\mathrm{SE}}\Phi^e(t_2;a)$ for all $a\in\mathcal{S}$ and all $0\leq t_1\leq t_2$. ■

B. Computing Robustly Forward Invariant Sets for Mixed-Monotone Systems

In the following theorem, we show how robustly forward invariant sets and attractive sets for (2) are efficiently computed via the identification of equilibria for the deterministic embedding system (9).

Theorem 1. *Suppose* (2) *is mixed-monotone with respect to* d*. Then for all* $a\in\mathcal{S}$ *the following hold:*

1) *For all* $t\geq 0$*, the set* $[\![\Phi^e(t;a)]\!]\subseteq\mathcal{X}$ *is robustly forward invariant for* (2)*.*
2) $\lim_{t\to\infty}\Phi^e(t;a) =: (x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}})$ *exists and* $e(x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}) = 0$, i.e., $(x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}})$ *is an equilibrium for* (9)*.*
3) *The set* $[x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}]$ *is robustly forward invariant and attractive from* $[\![a]\!]\subseteq\mathcal{X}$*.*

Proof. Part 1. Suppose $\mathcal{S}$ is nonempty, and choose $a\in\mathcal{S}$. Then, from Lemma 2, $\Phi^e(t;a)\in\mathcal{S}$ for all $t\geq 0$. Choose $\tau\geq 0$ and let $b=\Phi^e(\tau;a)\in\mathcal{S}$. Also from Lemma 2, $b\preceq_{\mathrm{SE}}\Phi^e(t;b)$ so that $[\![\Phi^e(t;b)]\!]\subseteq[\![b]\!]$ by (1) for all $t\geq 0$. From Proposition 2 we have $R^F(t;[\![b]\!])\subseteq[\![\Phi^e(t;b)]\!]$. Therefore $R^F(t;[\![b]\!])\subseteq[\![b]\!]$ for all $t\geq 0$, *i.e.*, $[\![b]\!]$ is robustly forward invariant for (2). This completes the proof of the first part since $\tau\geq 0$ was chosen arbitrarily.

Part 2. This result is a direct result of [25, Theorem 2.1]. In particular, since $a\preceq_{\mathrm{SE}}\Phi^e(t;a)$ for all $t\geq 0$, and $\mathcal{T}$ is forward on (9) we have

$$\Phi^e(t;a)\in\{(\underline{b},\overline{b})\in\mathcal{X}\times\mathcal{X}\ \mid\ \underline{a}\preceq\underline{b}\preceq\overline{b}\preceq\overline{a}\} \tag{21}$$

for all $t\geq 0$, where we define $\underline{a},\overline{a}\in\mathcal{X}$ by $a=(\underline{a},\overline{a})$. Thus $\lim_{t\to\infty}\Phi^e(t;a) := (x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}})$ exists and $e(x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}) = 0$.

Part 3. Choose $x\in[\![a]\!]$ and $\mathbf{w}:[0,\infty]\to\mathcal{W}$. Then

$$(\Phi^F(t;x,\mathbf{w}),\Phi^F(t;x,\mathbf{w})) = \Phi^\varepsilon(t;(x,x),(\mathbf{w},\mathbf{w})),$$
$$\Phi^e(t;a) = \Phi^\varepsilon(t;a,(\underline{w},\overline{w}))$$

hold for all $t\geq 0$. Since $a\preceq_{\mathrm{SE}}(x,x)$ and $(\underline{w},\overline{w}))\preceq_{\mathrm{SE}}(\mathbf{w}(t),\mathbf{w}(t))$ for all $t\geq 0$, we now have $\Phi^F(t;x,\mathbf{w})\in[\![\Phi^e(t;a)]\!]$ for all $t\geq 0$. Choose a relatively open neighborhood $\mathcal{X}_\epsilon$ of $[x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}]$ and a relatively open ball $B\subset\mathcal{X}\times\mathcal{X}$ such that $(x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}})\in B\subset\mathcal{X}_\epsilon\times\mathcal{X}_\epsilon$. From Part 2, there must exist a $\tau\geq 0$ such that $\Phi^e(\tau;a)\in B$ and at this time $\Phi^F(\tau;x,\mathbf{w})\in\mathcal{X}_\epsilon$. From Part 1 we have that $[\![\Phi^e(\tau;a)]\!]$ is robustly forward invariant for (2) and therefore $\Phi^F(t;x,\mathbf{w})\in\mathcal{X}_\epsilon$ for all $t\geq\tau$. Therefore, $[x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}]$ is attractive on (2) from $[\![a]\!]$. The fact that $[x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}]$ is robustly forward invariant follows immediately from Part 1. □

Theorem 1 provides a basic algorithm for identifying invariant sets for (2); if (9) has an *equilibrium*, *i.e.*, there exists an $(x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}})\in\mathcal{T}$ such that $e(x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}) = 0$, then

$$\mathcal{X}_{\mathrm{eq}} := [x_{\mathrm{eq}},\widehat{x}_{\mathrm{eq}}] \tag{22}$$

is robustly forward invariant for (2). Computing equilibria for (9) requires one to solve a system of $2n$ nonlinear equations and, therefore, does not require excessive computation. Moreover, if a point $a\in\mathcal{S}$ is known, then one can simulate the embedding dynamics forward in time, starting from a, in order to find an equilibria; see Theorem 1 Part 2.

In the following corollary, we show how globally attractive regions for (2) are identified using stability analysis in the embedding space.

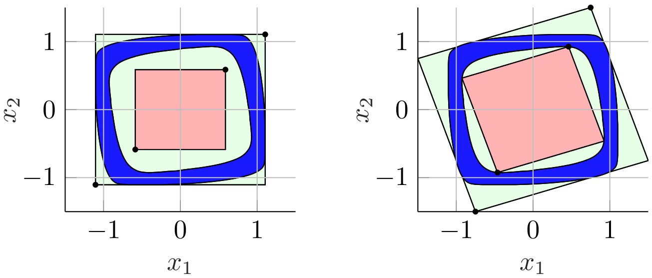
Corollary 1. Suppose (2) is mixed-monotone with respect to d. If $a\in\mathcal{T}$ is an asymptotically stable equilibrium for (9) with a basin of attraction $\mathcal{C}\subseteq\mathcal{X}\times\mathcal{X}$, then $[\![a]\!]$ is robustly forward invariant for (2) and attractive from all $[\![b]\!]$ such that $b\in\mathcal{C}\cap\mathcal{T}$. In particular, if $\mathcal{C}\supseteq\mathcal{T}$, then $[\![a]\!]$ is globally attractive and robustly forward invariant for (2). ■

Proof. Robust forward invariance of $[\![a]\!]$ follows immediately from Theorem 1, Part 1. Attractivity of $[\![a]\!]$ follows by a slight modification of the proof of Theorem 1, Part 3, where we observe that Part 2 of the theorem is invoked to establish that $\Phi^e(t;b)\in B$ for some $t\geq 0$, but this now holds for all $b\in\mathcal{C}$. Thus $\lim_{t\to\infty}\Phi^e(t;b) = a\in B$. □

We next extend Theorem 1 to leverage the backward-time dynamics (13). Specifically, we show that if (13) is mixed-monotone—a property explored previously in Remark 2—then robustly forward invariant sets for (2) can be computed using a technique analogous to Theorem 1.

Theorem 2. *Let* (13) *be mixed-monotone with respect to* D*. If there exists* $(\underline{x},\overline{x})\in\mathcal{T}$ *so that* $0\preceq_{\mathrm{SE}}E(\underline{x},\overline{x})$ *then* $\mathcal{X}\backslash[\underline{x},\overline{x}]$ *is robustly forward invariant for* (2)*, where* E *is the embedding function constructed from* D *as in* (9)*.*

We demonstrate the applicability of Theorem 1 for computing forward invariant regions in the following example.



(a) (b)

Fig. 2: Example 2: computing robustly forward invariant sets for (23) by applying Theorems 1 and 2. Subfigures: (a) $\mathcal{X}_{\text{eq}}$ is the larger rectangle and $\mathcal{Y}_{\text{eq}}$ is the smaller rectangle. It follows from Theorems 1 and 2 that the shaded green region is robustly forward invariant and attractive for (23), and the complement of the red region is forward invariant for (23). The region shown in blue is the smallest attractive set computed numerically. (b) The larger parallelotope corresponds to an equilibrium in the embedding space of (26), and the smaller parallelotope corresponds to an equilibrium in the embedding space of the backward-time dynamics. The shaded green region is robustly forward invariant and attractive for (23), and the complement of the red region is also forward invariant.

Example 2. We consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = F(x,w) = \begin{bmatrix}-x_2 + x_1(4 - 4x_1^2 - x_2^2) + w_1\\ x_1 + x_2(4 - x_1^2 - 4x_2^2) + w_2\end{bmatrix} \quad (23)$$

with $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{W} = [-3/4, 3/4] \times [-3/4, 3/4]$. A tight decomposition function is formed for (23), and a globally asymptotically stable equilibrium for (9) is identified using Theorem 1 part 2. In particular, we find $e(x_{\text{eq}}, \widehat{x}_{\text{eq}}) = 0$ for

$$x_{\text{eq}} = (-1.1, -1.1), \qquad \widehat{x}_{\text{eq}} = (1.1, 1.1), \quad (24)$$

and, therefore, the hyperrectangle $\mathcal{X}_{\text{eq}} := [x_{\text{eq}}, \widehat{x}_{\text{eq}}]$ is robustly forward invariant and globally attractive for (23).

Next, a tight decomposition function for the backward-time dynamics (13) is formed and $E(y_{\text{eq}}, \widehat{y}_{\text{eq}}) = 0$ for

$$y_{\text{eq}} = (-0.59, -0.59), \qquad \widehat{y}_{\text{eq}} = (0.59, 0.59), \quad (25)$$

where E denotes the embedding function of the backward-time dynamics as prescribed in Theorem 2. Thus, the set $\mathcal{X} \setminus \mathcal{Y}_{\text{eq}}$ is robustly forward invariant for (23), where $\mathcal{Y}_{\text{eq}} := [y_{\text{eq}}, \widehat{y}_{\text{eq}}]$. We show $\mathcal{X}_{\text{eq}}$ and $\mathcal{Y}_{\text{eq}}$ graphically in Figure 2a.

The basic procedures discussed in this work can be extended using the results of [20] for the computation of parallelotope invariant and attractive sets in a similar way. An example is shown in Figure 2b, where we employ a decomposition function for the transformed system

$$\dot{x} = T^{-1}F(Tx, w) \;\text{ with }\; T = \begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix} \quad (26)$$

to compute parallelotope invariant sets for (23). Further discussion on identifying suitable transformation matrices is provided in Section V-A of this work. ■

C. Necessary Conditions for Robust Forward Invariance

Theorem 1 provides a sufficient condition for the existence of robustly forward invariant sets for (2): given a state $(x, \widehat{x}) \in \mathcal{S}$, where $\mathcal{S}$ is computed from the embedding system according to (20), the hyperrectangular set $[x, \widehat{x}] \subseteq \mathcal{X}$ is robustly forward invariant for (2). Note, however, that the set $\mathcal{S}$ from (20) is dependent on the choice of decomposition function for (2) via the embedding dynamics (9), and certain decomposition functions will lead to more restrictive invariant sets than others. Thus, even when $(x, \widehat{x}) \notin \mathcal{S}$, the hyperrectangular set $[x, \widehat{x}] \subseteq \mathcal{X}$ may still be forward invariant for (2). To address this point, we next present a necessary condition for the existence of robustly forward invariant sets for (2).

Proposition 3. *Suppose* (2) *is mixed-monotone with a tight decomposition function* d *satisfying* (4)*. Then the hyperrectangular set* $[\underline{x}, \overline{x}] \subseteq \mathcal{X}$ *is robustly forward invariant for* (2) *if and only if* $(\underline{x}, \overline{x}) \in \mathcal{S}$*, where* $\mathcal{S}$ *is as given in* (20) *with* e *obtained from the tight decomposition function* d*.*

Proof. (if) It follows from Theorem 1 that $[\underline{x}, \overline{x}] \subseteq \mathcal{X}$ is robustly forward invariant for (2) whenever $(\underline{x}, \overline{x}) \in \mathcal{S}$.

(only if) Assume there exists a $(\underline{x}, \overline{x}) \in \mathcal{T}$ so that $[\underline{x}, \overline{x}] \subseteq \mathcal{X}$ is robustly forward invariant for (2). Then

$$\max_{\substack{y \in [\underline{x}, \overline{x}]\\ y_i = \overline{x}_i\\ z \in [w, \widehat{w}]}} F_i(y, z) \leq 0 \leq \min_{\substack{y \in [\underline{x}, \overline{x}]\\ y_i = \underline{x}_i\\ z \in [w, \widehat{w}]}} F_i(y, z) \quad (27)$$

must hold for all $i \in \{1, \cdots, n\}$. To see this, suppose the first inequality does not hold for some i so that there exists a $y \in [\underline{x}, \overline{x}]$ with $y_i = \underline{x}_i$ and a $z \in \mathcal{W}$ so that $F_i(y, z) > 0$. Then, initializing at y subject to constant disturbance z, the system will leave the forward set $[\underline{x}, \overline{x}]$ for sufficiently short horizon; *i.e.*, $\Phi_i^F(t; y, z) > \overline{x}_i$ so that $\Phi^F(t; y, z) \notin [\underline{x}, \overline{x}]$ for t small enough. A similar argument applies if the second inequality of (27) does not hold for some i. Equivalently

$$d(\overline{x}, \overline{w}, \underline{x}, \underline{w}) \preceq 0 \preceq d(\underline{x}, \underline{w}, \overline{x}, \overline{w}), \quad (28)$$

where d is the tight decomposition for (2) as defined by (4), *i.e.*, $(\underline{x}, \overline{x}) \in \mathcal{S}$. Therefore $[\underline{x}, \overline{x}] \subseteq \mathcal{X}$ is robustly forward invariant for (2) if and only if $(\underline{x}, \overline{x}) \in \mathcal{S}$. □

D. Invariant Sets from Periodic Orbits

Our next result is to show how invariant regions for (2) are identified from periodic orbits for the embedding system (9).

Theorem 3. *Assume for some* $a \in \mathcal{T}$ *there exist a nontrivial periodic solution* $\Phi^e(\,\cdot\,; a)$ *to* (9)*, i.e., there exists a period* $\tau > 0$ *so that* $\Phi^e(t + \tau; a) = \Phi^e(t; a)$ *holds for all* $t \geq 0$*. Then the set*

$$\Gamma(a, \tau) = \bigcup_{t \in [0, \tau]} [\![\Phi^e(t; a)]\!] \quad (29)$$

is robustly forward invariant for (2)*.*

Proof. Choose $a \in \mathcal{T}$ and assume there exists a $\tau > 0$ so that $\Phi^e(t + \tau; a) = \Phi^e(t; a)$ holds for all $t \geq 0$. Define $\Gamma(a, \tau)$ by (29) and choose $x \in \Gamma(a, \tau)$. Then there exist a $t^* \in [0, \tau]$

so that $x \in [\![\Phi^e(t^*;a)]\!]$ or equivalently $\Phi^e(t^*;a) \preceq_{\text{SE}} (x,x)$. Therefore, for all $t \geq 0$

$$R(t;x) \subseteq [\![\Phi^e(t;(x,x))]\!] \subseteq [\![\Phi^e(t+t^*;a)]\!] \subseteq \Gamma(a,\tau), \quad (30)$$

and, thus, $\Gamma(a,\tau)$ is robustly forward invariant for (2). □

Remark 3. Using analogous reasoning to that provided in Theorem 1 Part 3, it can additionally be shown that if $\{\Phi^e(t;a) \,|\, t \in [0,\tau]\}$ is attractive from $\mathcal{C} \subseteq \mathcal{X} \times \mathcal{X}$ then $\Gamma(a,\tau)$ is attractive from all $[\![b]\!]$ such that $b \in \mathcal{C} \cap \mathcal{T}$. We demonstrate this assertion through example in Section V-B where we compute a robustly forward invariant and attractive region for a 4-dimensional nondeterministic system via the identification of an attractive periodic orbit for the embedding system (9). ■

V. Case Studies

In this section, we demonstrate the applicability of Theorems 1 and 3 through two concluding numerical examples.

A. Case Study 1: Invariance for Planar Quadrotor

Consider a model for a planar quadrotor, *i.e.*, a quadrotor fixed in the X-Z plane, with dynamics

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix} = F(x,u) = \begin{bmatrix} x_3 \\ x_4 \\ -u_1\sin(x_5) \\ u_1\cos(x_5) - g \\ u_2 \end{bmatrix} \quad (31)$$

where x_1 and x_2 are the horizontal and vertical positions of the quadrotor, x_3 and x_4 are the respective velocities, x_5 is the pitch angle, $u = (u_1, u_1) \in \mathbb{R}^2$ is a control input with u_1 the net thrust in the direction that the quadrotor is oriented and u_2 the roll angular velocity, and $g = 9.81$ is the gravitational constant. Denote also by

$$\dot{x} = F(x) := F(x, u(x)) \quad (32)$$

the closed-loop dynamics of (31) under the feedback controller

$$u(x) = \begin{bmatrix} 0 & -40 & 0 & -20 & 0 \\ 3 & 0 & 6 & 0 & -25 \end{bmatrix} x + \begin{bmatrix} g \\ 0 \end{bmatrix}, \quad (33)$$

and observe that the origin is a globally asymptotically stable equilibrium for (32).

As in Example 2, we compute a forward invariant region for (31)–(33) by considering a linear transformation of the dynamics. To construct a suitable transformation, we consider a local linearization of (31)–(33) at the origin,

$$\dot{x} = Ax \qquad \text{with} \qquad A := \frac{\partial F}{\partial x}\Big|_{x=0}. \quad (34)$$

The matrix A has only negative real eigenvalues and, thus, the origin is a locally stable equilibrium for (31)–(33). We consider the transformed system

$$\dot{x} = TF(T^{-1}x), \quad (35)$$

where the columns of $T \in \mathbb{R}^{5\times 5}$ are the eigenvectors of A.

We omit the decomposition function for the transformed system (35) due to space constraints, but we note that a key element in deriving the decomposition function comes from decomposing the auxiliary system

$$\dot{x} = w_1 \sin(w_2), \quad (36)$$

which is mixed-monotone with respect to

$$d'(x,\widehat{x}) = \begin{cases} \min\{A\} + \min\{B\} & \text{if } x \preceq \widehat{x} \\ \max\{A\} + \max\{B\} & \text{if } \widehat{x} \preceq x \end{cases} \quad (37)$$

where w is viewed as a disturbance and where

$$\begin{aligned} A &:= \{y_1\sin(y_2) + y_1 y_2 \mid y \in \{w_1, \widehat{w}_1\} \times \{w_2, \widehat{w}_2\}\} \\ B &:= \{-y_1 y_2 \mid y \in \{w_1, \widehat{w}_1\} \times \{w_2, \widehat{w}_2\}\}. \end{aligned} \quad (38)$$

In particular, a decomposition function for (35) is formed as a weighted sum of decomposition functions of the form (37).

We then apply Theorem 1 with the decomposition function for (35) to identify robustly forward invariant sets for (32). We find $0 \preceq_{\text{SE}} e(\underline{x}, \overline{x})$ for

$$\overline{x} = -\underline{x} = (0,0,0,1,3.61). \quad (39)$$

Therefore $[\underline{x}, \overline{x}] \subset \mathbb{R}^5$ is robustly forward invariant for (35) and

$$[\underline{x}, \overline{x}]_T := \{x \in \mathbb{R}^5 \mid T^{-1}x \in [\underline{x}, \overline{x}]\} \subset \mathcal{X} \quad (40)$$

is forward invariant for (32). The invariant set derived in this study is shown graphically in Figure 3.

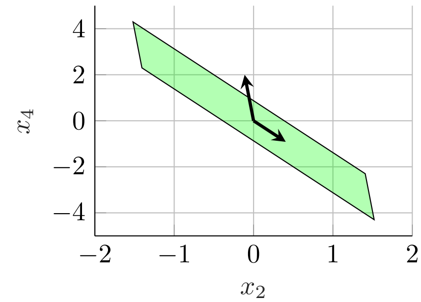


Fig. 3: Case Study 1: invariant regions for the quadrotor model (31) under (33), projected onto the x_2, x_4 plane. The parallelogram $[\underline{x}, \overline{x}]_T$, with $\underline{x}$ and $\overline{x}$ from (39) is shown in green. Since $0 \preceq_{\text{SE}} e(\underline{x}, \overline{x})$, it follows from Theorem 1 that the green region is forward invariant for (31) under (33). Two eigenvectors of A, which are used to construct the transformation T, are shown as black arrows.

B. Case Study 2

Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = F(x,w) = \begin{bmatrix} -2x_1 + x_2(1+x_1) + x_3 + w \\ -x_2 + x_1(1-x_2) + 0.1 \\ -x_4 \\ x_3 \end{bmatrix} \quad (41)$$

with $\mathcal{X} = \mathbb{R}^4$ and $\mathcal{W} = [-0.1, 0.1]$. The system is mixed-monotone with a tight decomposition function given in closed form, by

$$\begin{aligned} d_1(x,w,\widehat{x},\widehat{w}) &= -2x_1 + a(x,\widehat{x}) + x_3 + w \\ d_2(x,w,\widehat{x},\widehat{w}) &= -x_2 + b(x,\widehat{x}) + 0.1, \\ d_3(x,w,\widehat{x},\widehat{w}) &= -\widehat{x}_4 \\ d_4(x,w,\widehat{x},\widehat{w}) &= x_3, \end{aligned} \quad (42)$$

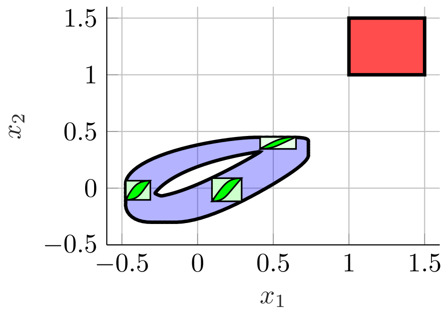


Fig. 4: Case Study 2: computing robustly forward invariant and attractive sets for (41) by applying Theorem 3. The robustly forward invariant and attractive set $\Gamma(a,\tau)$ is shown in blue. To demonstrate the attractiveness of $\Gamma(a,\tau)$, we compute reachable sets from $\mathcal{X}_0 = [1, 1.5]^2 \times \{1\} \times \{0\}$ using Proposition 2. $\mathcal{X}_0$ is shown in red. $R^F(10, \mathcal{X}_0)$, $R^F(12, \mathcal{X}_0)$ and $R^F(14, \mathcal{X}_0)$ are shown in green (left to right), with reachable set overapproximations derived from the application of Proposition 2 shown in light green.

where

$$a(x,\widehat{x}) := \begin{cases} \min\{x_2(1+x_1), \widehat{x}_2(1+x_1)\} & \text{if } x \preceq \widehat{x} \\ \max\{x_2(1+x_1), \widehat{x}_2(1+x_1)\} & \text{if } \widehat{x} \preceq x \end{cases}$$

$$b(x,\widehat{x}) := \begin{cases} \min\{x_1(1-x_2), \widehat{x}_1(1-x_2)\} & \text{if } x \preceq \widehat{x} \\ \max\{x_1(1-x_2), \widehat{x}_1(1-x_2)\} & \text{if } \widehat{x} \preceq x \end{cases}$$

and a periodic orbit is identified in the resulting embedding system. We find $\Phi^e(t+\tau; a) = \Phi^e(t; a)$ for

$$a = (-0.03, -0.20, 0.68, -0.75, 0.16, 0.02, 0.68, -0.75) \tag{43}$$

and $\tau = 2\pi$. Thus, it follows from Theorem 3 that $\Gamma(a,\tau)$ is robustly forward invariant for (41), where $\Gamma(a,\tau)$ is given by (29). Moreover, $\{\Phi^e(t;a) \,|\, t \in [0,\tau]\}$ is attractive from

$$\mathcal{C} = \{(x,\widehat{x}) \in \mathcal{T} \,|\, x_3 = \widehat{x}_3, x_4 = \widehat{x}_4, x_3^2 + x_4^2 = 1\} \tag{44}$$

and, therefore, $\Gamma(a,\tau)$ is attractive from

$$\{x \in \mathcal{X} \,|\, x_3^2 + x_4^2 = 1\} \subset \mathcal{X}. \tag{45}$$

The robustly forward invariant and attractive set $\Gamma(a,\tau)$ derived in this study is shown graphically in Figure 4.

VI. Conclusion

This work presents new tools for studying reachability and set invariance for nondeterministic systems subject to a disturbance input using the theory of mixed-monotone dynamical systems. Our main contribution is an efficient method for identifying robustly forward invariant and attractive sets for mixed-monotone systems, and our contributions specifically enable control system safety applications where, often, precise knowledge of robustly forward invariant sets is required [17], [18]. The findings of this work are demonstrated through two numerical examples and a two concluding case studies where we compute forward invariant regions for, *e.g.*, a 5-dimensional planar quadrotor system.

References

[1] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[2] S. Prajna and A. Jadbabaie, "Safety verification of hybrid systems using barrier certificates," in *Hybrid Systems: Computation and Control* (R. Alur and G. J. Pappas, eds.), (Berlin, Heidelberg), pp. 477–492, Springer Berlin Heidelberg, 2004.

[3] D. Angeli, G. A. Enciso, and E. D. Sontag, "A small-gain result for orthant-monotone systems under mixed feedback," *Systems & Control Letters*, vol. 68, pp. 9 – 19, 2014.

[4] G. Enciso, H. Smith, and E. Sontag, "Nonmonotone systems decomposable into monotone systems with negative feedback," *Journal of Differential Equations*, vol. 224, no. 1, pp. 205 – 227, 2006.

[5] L. Yang, O. Mickelin, and N. Ozay, "On sufficient conditions for mixed monotonicity," *IEEE Transactions on Automatic Control*, vol. 64, pp. 5080–5085, Dec 2019.

[6] S. Coogan and M. Arcak, "Stability of traffic flow networks with a polytree topology," *Automatica*, vol. 66, pp. 246–253, Apr. 2016.

[7] H. L. Smith, "The discrete dynamics of monotonically decomposable maps," *Journal of Mathematical Biology*, vol. 53, p. 747, May 2006.

[8] P. Meyer and D. V. Dimarogonas, "Hierarchical decomposition of LTL synthesis problem for nonlinear control systems," *IEEE Transactions on Automatic Control*, vol. 64, pp. 4676–4683, Nov 2019.

[9] P.-J. Meyer, A. Devonport, and M. Arcak, "Tira: Toolbox for interval reachability analysis," in *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, HSCC '19, p. 224–229, Association for Computing Machinery, 2019. An extended version of this work appears on arXiv `https://arxiv.org/abs/1902.05204`.

[10] S. Coogan and M. Arcak, "Efficient finite abstraction of mixed monotone systems," in *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, pp. 58–67, 2015.

[11] H. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. Mathematical surveys and monographs, American Mathematical Society, 2008.

[12] D. Angeli and E. D. Sontag, "Monotone control systems," *IEEE Transactions on Automatic Control*, vol. 48, pp. 1684–1698, Oct 2003.

[13] H. Smith, "Global stability for mixed monotone systems," *Journal of Difference Equations and Applications*, vol. 14, no. 10-11, pp. 1159–1164, 2008.

[14] T. Chu and L. Huang, "Mixed monotone decomposition of dynamical systems with application," *Chinese Science Bulletin*, vol. 43, pp. 1171–1175, Jul 1998.

[15] M. Dutreix and S. Coogan, "Efficient verification for stochastic mixed monotone systems," in *2018 ACM/IEEE 9th International Conference on Cyber-Physical Systems (ICCPS)*, pp. 150–161, April 2018.

[16] C. Santoyo, M. Dutreix, and S. Coogan, "Verification and control for finite-time safety of stochastic systems via barrier functions," in *2019 IEEE Conference on Control Technology and Applications (CCTA)*, pp. 712–717, Aug 2019.

[17] M. Abate and S. Coogan, "Enforcing safety at runtime for systems with disturbances," in *2020 59th IEEE Conference on Decision and Control (CDC)*, pp. 2038–2043, 2020.

[18] M. Abate, M. Mote, E. Feron, and S. Coogan, *Verification and Runtime Assurance for Dynamical Systems with Uncertainty*. New York, NY, USA: Association for Computing Machinery, 2021.

[19] M. Abate and S. Coogan, "Computing robustly forward invariant sets for mixed-monotone systems," in *2020 59th IEEE Conference on Decision and Control (CDC)*, pp. 4553–4559, 2020.

[20] M. Abate and S. Coogan, "Improving the fidelity of mixed-monotone reachable set approximations via state transformations," in *2021 American Control Conference (ACC)*, pp. 4674–4679, 2021.

[21] C. G. Mayhew and A. R. Teel, "On the topological structure of attraction basins for differential inclusions," *Systems & Control Letters*, vol. 60, no. 12, pp. 1045 – 1050, 2011.

[22] S. Coogan, "Mixed monotonicity for reachability and safety in dynamical systems," in *2020 59th IEEE Conference on Decision and Control (CDC)*, pp. 5074–5085, 2020.

[23] M. Abate, M. Dutreix, and S. Coogan, "Tight decomposition functions for continuous-time mixed-monotone systems with disturbances," *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 139–144, 2021.

[24] M. Khajenejad and S. Z. Yong, "Tight remainder-form decomposition functions with applications to constrained reachability and interval observer design," 2021.

[25] M. W. Hirsch, "Differential equations and convergence almost everywhere in strongly monotone semiflows," *Contemp. Math*, vol. 17, pp. 267–285, 1983.