Formation Control with Size Scaling using Relative Displacement Feedback

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***Abstract*— We study a multiagent formation control problem where the objective is to steer a team of double integrator agents evolving in Euclidean space of arbitrary dimension into a desired formation. The size scaling of the desired formation is known only to a subset of the agents, and we present two decentralized strategies by which the agents can accomplish the formation task using only relative displacement feedback. A variable formation size allows the team of agents to adjust to changes in the environment or in group objectives, or respond to perceived threats, and control laws using only relative displacement information enables these strategies to be implemented using only local sensors and no interagent communication. We demonstrate the proposed strategies through several examples with simulations.**

I. Introduction

Group coordination of multiple agents using distributed feedback laws has received considerable research attention in recent years as implementation costs have decreased and the results are suitable to a wide range of applications. The main emphasis of such coordination problems is to achieve a desired group behavior using only local feedback rules, [1], [2], [3], [4], [5], [6], [7]. Formation control and maintenance is a common group task that can be achieved using local feedback strategies and has been addressed using a variety of methods, e.g. [8], [9], [10], [11], and [12].

In this paper, we consider the problem of formation control when only a subset of the agents, henceforth called the leaders, know the desired formation size. The remaining follower agents implement a cooperative control law using only local interagent position information such that the agents converge to the desired formation scaled by the desired size. By allowing the size of the formation to change, the group can dynamically adapt to changes in the environment such as unforeseen obstacles, adapt to changes in group objectives, or respond to threats.

We adopt a standard control scheme for the leaders and propose two cooperative control strategies for the follower agents. In both designs, we assume the agents are equipped with sensing capabilities such that the relative displacements of neighboring agents are available to an agent, thus the proposed control laws are distributed and can be implemented without the need for direct interagent communication. Such a construction may be valuable in scenarios where communication is expensive or dangerous. In addition, it is of theoretical interest in cooperative control problems to understand what can be accomplished with only relative position feedback.

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Size scaling of a formation in the presence of direct interagent communication with only one leader was studied in [13]. A special case of the single link method presented in this paper in which each team only has one leader and the follower agents use feedback with memory was also studied in [13]. In the present sequel, we allow multiple leaders, we consider the case of static feedback, and we present an alternative feedback method in which each agent monitors multiple links for the purpose of estimating the formation scale.

This paper is organized as follows: Section II introduces the problem statement. In Sections III and IV, we present two control methods, referred to as the *single link* method and the *multiple link* method, to solve the formation control problem. Section V presents several simulations, and Section VI summarizes our main results.

II. Preliminaries

A. Problem Definition

We consider a team of n mobile agents evolving in $\mathbb{R}^p$. We represent the position of each agent by a vector $x_i(t) \in \mathbb{R}^p$ for $i = 1, \dots, n$ and model the agents as double integrators, *i.e.*

$$\ddot{x}_i(t) = u_i(t). \tag{1}$$

Suppose agents are capable of sensing the relative displacement of neighboring agents. We assume relative position sensing is bidirectional and define a time-invariant position sensing topology using an undirected *sensing graph* with n nodes where an edge exists between nodes i and j if and only if agents i and j have access to the relative position vector $x_i(t) - x_j(t)$. We assume the sensing graph is connected.

Suppose there are m edges in the sensing graph. To simplify analysis, we arbitrarily assign a direction to each edge of the graph. Note that, because the sensing topology is bidirectional, the choice of direction does not affect the following results. We now define the relative displacement vector associated with edge k with head node i and tail node j as

$$z_k(t) \triangleq x_i(t) - x_j(t). \tag{2}$$

A formation is defined as a set of m desired displacement vectors, $\{z_j^d\}_{j=1}^m \subset \mathbb{R}^p$. We will assume $||z_j^d|| > 0$ for all j. In addition, we wish to allow the size of the formation to vary, thus we define a desired formation scale $\lambda \in \mathbb{R}$ and the scaled formation is defined as the set $\{z_j^d \lambda\}_{j=1}^m$. Note that, while $\lambda \leq 0$ may not be practically desirable depending on the particular application, it nonetheless results in a well-defined scaled formation and does not alter the theoretical results or physical interpretations presented in this paper.

Remark 1. To prove stability and convergence results, we assume λ is static. However, it will be clear that if λ is allowed to vary with time, and if λ does not vary quickly or often with respect to the speed of the agent dynamics, then the following strategies will approximately track a size-varying formation. This is demonstrated in the simulations.

We assume a subset of the agents have *a priori* knowledge of λ, and we refer to these agents as *leaders* and the remaining agents as *followers*. Our goal is to design $u_i(t)$ for $i = 1, \dots, n$ to achieve the following group behavior:

$$\lim_{t\to\infty} z_k(t) = z_k^d \lambda \quad \text{for } k = 1, \dots, m, \tag{3}$$

$$\lim_{t\to\infty} v_i(t) = 0 \quad \text{for } i = 1, \dots, n \tag{4}$$

where $v_i(t) \triangleq \dot{x}_i(t)$.

All agents have access to the formation specification $\{z_j^d\}_{j=1}^m$, but only the leaders have access to λ, thus we seek a control strategy for the leaders and the followers that achieves (3) and (4), relies only on local relative position information, and does not depend explicitly on λ in the case of the follower agents.

B. Notation

Throughout the remainder of this paper, explicit dependence on t is suppressed for time-varying quantities. $\mathbf{1}_k$ is the length k column vector of all ones, $\mathbf{0}_k$ is the length k column vector of all zeros, I_p is the $p\times p$ identity matrix, and $0_{k\times l}$ is the zero matrix of dimension $k\times l$. When dimensions are clear, subscripts are omitted. The operator BlockDiag$\{\cdot\}$ returns a block diagonal matrix with its arguments along the block diagonal.

We define the stacked vectors $x \triangleq \begin{bmatrix} x_1^T & \dots & x_n^T \end{bmatrix}^T$, $v \triangleq \dot{x}$, $z \triangleq \begin{bmatrix} z_1{}^T & \dots & z_m{}^T \end{bmatrix}^T$ and $z^d \triangleq \begin{bmatrix} (z_1^d)^T & \dots & (z_m^d)^T \end{bmatrix}^T$.

Unless explicitly stated, we assume the subset of leader agents is nonempty and denote the number of leaders by n_l and the number of followers by n_f. For ease of analysis, we index the followers first, thus the set $\mathcal{I}_F \triangleq \{1, \dots, n_f\}$ is the set of indices corresponding to the follower agents and the set $\mathcal{I}_L \triangleq \{n_f+1, \dots, n\}$ is the set of indices corresponding to the leader agents.

We now define the $n\times m$ incidence matrix D element-wise as follows:

$$d_{ij} = \begin{cases} +1 & \text{if node } i \text{ is the head of edge } j \\ -1 & \text{if node } i \text{ is the tail of edge } j \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that D has rank $n-1$ when the sensing graph is connected, has linearly independent columns only when there are no cycles in the graph, and that $z = (D^T \otimes I_p)x$ where $\otimes$ is the Kronecker product. Thus we require $z^d \in \mathcal{R}(D^T \otimes I_p)$ in order for the formation to be well-defined (*i.e.*, there exist agent positions in $\mathbb{R}^p$ such that $z = z^d$).

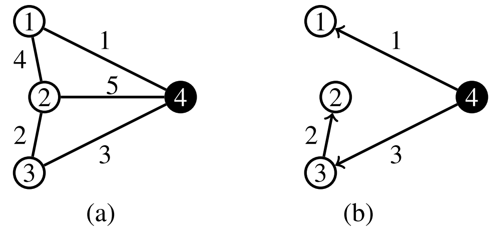


Fig. 1. (a) 4-agent, 5-edge sensing graph where agent 4 is the leader with edge indexing based on the monitoring graph. (b) One possible monitoring graph constructed from the sensing graph.

We partition D into the rows corresponding to the leader agents and rows corresponding to the follower agents as follows:

$$\begin{bmatrix} D_f \\ D_l \end{bmatrix} \triangleq D \tag{5}$$

where D_f consists of the first n_f rows of D, and D_l consists of the remaining rows.

C. Leader Strategy

For the leaders, the following is a standard strategy that will be employed for both the single link and the multiple link methods:

$$u_i = -\sum_{j=1}^{m} d_{ij}(z_j - z_j^d \lambda) - k v_i, \quad i \in \mathcal{I}_L \tag{6}$$

where $k > 0$ is a damping coefficient.

III. Single Link Method

We assume the leader agents employ the feedback strategy (6). Observe that this method is not suitable for the follower agents, as it requires knowledge of λ. To develop a method for estimating λ to be used by the follower agents, we consider a directed subgraph of the sensing graph with the following properties:

- The subgraph contains all the vertices and a (directed) subset of the edges of the sensing graph.
- Each follower node is the head of exactly one edge. The leaders are the head of no edge.
- A directed path exists to each follower node originating from a leader node.

We call this subgraph the *monitoring graph*, and it is easy to see that the existence of such a subgraph is guaranteed by the connectivity of the undirected sensing graph. In the case of one leader, the monitoring subgraph is simply a directed spanning tree rooted at the leader. We have assumed the follower nodes are indexed first, and for notational purposes we now index the edges of the sensing graph such that for each edge of the monitoring graph, the index of the corresponding edge in the sensing graph matches the head node index from the monitoring graph. We arbitrarily index the remaining edges of the sensing graph. Thus vectors $z_1, \dots, z_{n_f}$ correspond to the monitoring edges of agents $1, \dots, n_f$, respectively. Fig. 1 shows an example of a formation with the sensing graph, one possible construction of the monitoring graph, and the induced edge indexing scheme.

Via this construction, we now define the following parameters:

$$\lambda_i \triangleq \frac{1}{||z_i^d||^2}(z_i^d)^T z_i, \quad i \in \mathcal{I}_f. \tag{7}$$

It is clear that if $z = z^d\lambda$, *i.e.* the desired formation has been achieved, then $\lambda_i = \lambda$ for all i. Furthermore, (7) can be viewed as the linearization of $||z_i||^2/||z_i^d||^2$ around the point $z_i = z_i^d\lambda$, motivating the following proposed update rule for the follower agents:

$$u_i = -\sum_{j=1}^{m} d_{ij}\left(z_j - z_j^d\lambda_i\right) - kv_i, \quad i \in \mathcal{I}_F \tag{8}$$

with λ_i defined as in (7).

To simplify notation, let

$$\Delta \triangleq \text{BlockDiag}\,\{\textstyle\sum_{j=1}^m d_{1j}\left|\left|z_1^d\right|\right|^{-2} z_j^d (z_1^d)^T, \ldots, \sum_{j=1}^m d_{n_f j}\left|\left|z_{n_f}^d\right|\right|^{-2} z_j^d (z_{n_f}^d)^T\} \tag{9}$$

and

$$\tilde{z} \triangleq z - z^d\lambda. \tag{10}$$

We then write the strategies (6) and (8) in matrix form as

$$\dot{v} = -kv - (D \otimes I_p)\tilde{z} + \begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}\tilde{z} \tag{11}$$

where we have used the fact $(D_f \otimes I_p)z^d = \begin{bmatrix}\Delta & 0\end{bmatrix} z^d$.

We consider stability of the origin of $\begin{bmatrix}v^T & \tilde{z}^T\end{bmatrix}^T$ where the dynamics of the system are given by (11) and

$$\dot{\tilde{z}} = (D^T \otimes I_p)v. \tag{12}$$

Since $\mathcal{R}(D^T \otimes I_p)$ is invariant under the dynamics of (12) and $z^d \in \mathcal{R}(D^T \otimes I_p)$ by assumption, we have that the system (11)–(12) evolves in the following invariant subspace of $\mathbb{R}^{(n+m)p}$:

$$\mathcal{S} = \{(v, \tilde{z}) : v \in \mathbb{R}^{np}, \tilde{z} \in \mathcal{R}(D^T \otimes I_p)\}. \tag{13}$$

While the stability of the overall dynamics governed by (11)–(12) can be determined using a number of techniques including direct calculation of eigenvalues, we propose a simple condition derived via application of the small-gain theorem as a certificate of stability. While this condition may be conservative, it provides an appealing geometric criterion that is easy to check using the formation geometry and eigenvalues of the graph Laplacian matrix. To this end, we consider (11)–(12) as the interconnection of two subsystems:

1) Subsystem 1:

$$\dot{v} = -kv - (D \otimes I_p)\tilde{z} + g \tag{14}$$
$$\dot{\tilde{z}} = (D^T \otimes I_p)v \tag{15}$$
$$y = \tilde{z}. \tag{16}$$

2) Subsystem 2:

$$g = \begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix} y. \tag{17}$$

We have the following lemmas:

Lemma 1. *Let μ_1 denote the smallest positive eigenvalue of DD^T and μ_{n-1} the largest. For $k \geq \sqrt{2\mu_{n-1}}$, the $\mathcal{L}_2$ gain from g to y of subsystem 1 defined by* (14)–(16) *is*

$$\gamma_1 = \frac{1}{\sqrt{\mu_1}}. \tag{18}$$

Proof: *We observe that the "$\otimes I_p$" term only increases the multiplicity of a singular value and therefore it suffices to assume $p = 1$. Let $\{\mu_i\}_{i=0}^{n-1}$ be the eigenvalues of DD^T such that $0 = \mu_0 < \mu_1 \leq \cdots \leq \mu_{n-1}$.*

Let $D = U\Sigma V^T$ be the singular value decomposition of D with $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{m\times m}$, and $\Sigma \in \mathbb{R}^{n\times m}$ where $\Sigma = diag\{\{\sqrt{\mu_i}\}_{i_0}^{n-1}\}$. Note that if $m = n-1$, then 0 is not a singular value of D, thus we set $i_0 = 1$ if $m = n-1$ and $i_0 = 0$ otherwise.

We let $\bar{v} = U^T v$, $\bar{z} = V^T\tilde{z}$ and obtain the following decoupled system after a change of basis:

$$\dot{\bar{v}} = -k\bar{v} - \Sigma\bar{z} + \bar{g} \tag{19}$$
$$\dot{\bar{z}} = \Sigma^T\bar{v} \tag{20}$$
$$\bar{y} = \bar{z} \tag{21}$$

where $\bar{g} = U^T g$ and $\bar{y} = V^T y$. Each decoupled system with nonzero output takes the form

$$\dot{\bar{v}}_i = -k\bar{v}_i - \sqrt{\mu_i}\bar{z}_i + \bar{g}_i \tag{22}$$
$$\dot{\bar{z}}_i = \sqrt{\mu_i}\bar{v}_i \tag{23}$$
$$\bar{y}_i = \bar{z}_i \tag{24}$$

for $i = 1, \ldots, n-1$. Each of these subsystems has transfer function

$$G_i(s) = \frac{\sqrt{\mu_i}}{s^2 + ks + \mu_i}. \tag{25}$$

For $k \geq \sqrt{2\mu_i}$, the $\mathcal{L}_2$ gain of $G_i(s)$ is $1/\sqrt{\mu_i}$. Using the fact that U and V are orthonormal and therefore do not alter the magnitude of the input g or the output y, the lemma follows. ■

Lemma 2. *The $\mathcal{L}_2$ gain from y to g of subsystem 2 defined by* (17) *is*

$$\gamma_2 = \max_{i \in \mathcal{I}_F}\left\{\frac{\left|\left|\sum_{j=1}^m d_{ij} z_j^d\right|\right|}{||z_i^d||}\right\}. \tag{26}$$

Proof: Since $\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}$ is a static matrix, it is clear that

$$\gamma_2 = \left|\left|\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}\right|\right|_2 \tag{27}$$
$$= ||\Delta||_2 \tag{28}$$

which is simply the largest singular value of Δ. Since Δ is block diagonal, the singular values of Δ are just the singular values of the blocks. Each block is a dyad of the form

$$\left(\textstyle\sum_{j=1}^m d_{ij} z_j^d\right)\left(\frac{1}{||z_i^d||^2}(z_i^d)^T\right) \tag{29}$$

for $i \in \mathcal{I}_F$. Therefore each block has one nonzero singular value equal to the product of the norms of the column vector and the row vector which form the dyad, *i.e.* the nonzero singular values of Δ are

$$\left\{ \left|\left| \sum_{j=1}^{m} d_{ij} z_j^d \right|\right| \cdot \frac{1}{||z_i^d||} \right\}, \quad i \in \mathcal{I}_F. \tag{30}$$

■

From Lemma 2, we obtain a simple geometric procedure for determining the gain of the lower subsystem:

1) For each agent $i \in \mathcal{I}_F$, calculate $|| \sum_{j=1}^{m} d_{ij} z_j^d ||$, a quantity only dependent on the desired formation, and divide this by $||z_i^d||$.
2) The largest quotient is the gain of the lower subsystem, γ_2.

Theorem 1. *Let μ_1 be the smallest positive eigenvalue of DD^T and let μ_{n-1} be the largest eigenvalue. If $k \geq \sqrt{2\mu_{n-1}}$ and*

$$\frac{1}{\sqrt{\mu_1}} \cdot \max_{i \in \mathcal{I}_F} \left\{ \frac{\left|\left| \sum_{j=1}^{m} d_{ij} z_j^d \right|\right|}{||z_i^d||} \right\} < 1, \tag{31}$$

then the control strategy (6) *and* (8) *achieves the desired group behavior* (3) *and* (4).

Proof: Using Lemma 1, Lemma 2 and applying the small-gain theorem, see e.g. [14], we conclude that without exogenous input, both g and y are $\mathcal{L}_2$ functions. Because both subsystem 1 and subsystem 2 are linear, g and y are uniformly continuous, and, applying Barbalat's Lemma, we conclude $y \to 0$. ■

IV. Multiple Link Method

We again assume that the leader agents use strategy (6). We redefine

$$\lambda_j \triangleq \frac{1}{||z_j^d||^2} (z_j^d)^T z_j, \quad j = 1, \ldots, m, \tag{32}$$

this time associating a λ_j parameter with each edge and not just the monitoring edges as in (7). We propose the following strategy for the follower agents:

$$u_i = -\sum_{j=1}^{m} d_{ij} \left(z_j - z_j^d \lambda_j \right) - kv, \quad i \in \mathcal{I}_F. \tag{33}$$

Note that in (33), the indices of the λ_j parameters within the summation match the summation index and hence each λ_j is associated with a link and not an agent as in the single link method. The difference between (33) and (8) is that in (33), the estimate of λ changes for each link within the summation, whereas in (8), one link was assigned to each agent in order to estimate λ. Define

$$P_j \triangleq \frac{1}{||z_j^d||^2} z_j^d (z_j^d)^T \tag{34}$$

to be the orthogonal projection matrix onto

$$S_j \triangleq \text{span}\{z_j^d\} \subset \mathbb{R}^p, \tag{35}$$

and define

$$Q_j = I_p - P_j \tag{36}$$

to be the orthogonal projection onto the hyperplane $S_j^\perp$ where $^\perp$ denotes orthogonal complement. Also let

$$P \triangleq \text{BlockDiag}\{P_1, \ldots, P_m\} \tag{37}$$

$$Q \triangleq I_{pm} - P, \tag{38}$$

observing that P is the orthogonal projection matrix onto

$$S \triangleq S_1 \times \cdots \times S_m = \prod_{j=1}^{m} S_j \subset \mathbb{R}^{mp} \tag{39}$$

and Q is the orthogonal projection onto $S^\perp$. Define

$$D_Q \triangleq \begin{bmatrix} (D_f \otimes I_p)Q \\ (D_l \otimes I_p) \end{bmatrix}. \tag{40}$$

Then we can rewrite (33) as

$$u_i = -\sum_{j=1}^{m} d_{ij} Q_j z_j - k_i v_i, \quad i \in \mathcal{I}_F. \tag{41}$$

Observing that $Qz^d = 0$, we can combine the leader and follower strategies and write (6) and (33) in matrix form as

$$\dot{v} = -D_Q \tilde{z} - kv. \tag{42}$$

We are interested in the stability properties of the system defined by (42) and (12) which again evolves in the subspace (13).

We first explore the case in which we do not have any leaders, which is interesting in its own right and offers insight into the geometric requirements for ensuring that the scaled formation is attained.

Lemma 3. *Suppose there are no leader agents,* i.e. $D_Q = (D \otimes I_p)Q$. *Then strategy* (41) *ensures z asymptotically converges to a scaling of the desired formation z^d for all initial conditions if and only if*

$$\mathcal{R}(D^T \otimes I_P) \cap S = \textit{span}\{z^d\}. \tag{43}$$

Proof: Without a leader, we let $\lambda = 0$ and then $z = \tilde{z}$.

(if) Let $V \triangleq \frac{1}{2}(v^T v + z^T Q z)$ be a Lyapunov function for the system. Recalling that $\dot{z} = (D^T \otimes I_p)v$, we have $\dot{V} = -kv^T v \leq 0$. Applying LaSalle's principle, we see that $v \equiv 0$ only if $(D \otimes I_p)Qz = 0 \implies Qz \in \mathcal{N}(D \otimes I_p)$. But also $z \in \mathcal{R}(D^T \otimes I_p) = \mathcal{N}(D \otimes I_P)^\perp$. Therefore, if $v \equiv 0$, we have $||Qz||^2 = z^T Q^T Q z = z^T (Qz) = 0$, and thus $z \in \mathcal{N}(Q) = S$. By condition (43), we have $z \in \text{span}\{z^d\}$.

(only if) If we assume that condition (43) does not hold, then there exists a z^* such that $z^* \in \mathcal{R}(D^T \otimes I_p) \cap S$ and $z^* \notin \text{span}\{z^d\}$. Since $z^* \in S$ and Q is the orthogonal projection onto $S^\perp$, $Qz^* = 0$ and therefore $v = 0$, $z = z^*$ is an equilibrium of (41), but z^* is not a scaling of the desired formation. ■

A formation specification satisfying (43) is said to be *parallel rigid* and has been studied primarily in the computer-aided design literature, see *e.g.* [15]. There exists a duality between the concepts of parallel rigidity and the more standard distance-based rigidity, see [16] and [17] for details.

Theorem 2. *With at least one leader, the control strategy* (6) *for the leaders and the multiple link method* (33) *for the followers achieves the desired group behavior* (3) *and* (4) *for sufficiently large k if and only if the equilibrium subspace $\mathcal{R}(\mathbf{1}^T \otimes I_p)$ of the auxiliary system*

$$\dot{\xi} = -D_Q(D^T \otimes I_p)\xi \tag{44}$$

is asymptotically stable.

Proof: (if) We consider the asymptotic stability of the system defined by (42) and (12), which we write as

$$\begin{bmatrix} \dot{v} \\ \dot{\tilde{z}} \end{bmatrix} = A \begin{bmatrix} v \\ \tilde{z} \end{bmatrix} \tag{45}$$

with

$$A = \begin{bmatrix} -kI_{np} & -D_Q \\ (D^T \otimes I_p) & 0_{mp \times mp} \end{bmatrix}. \tag{46}$$

We have that (45) evolves in the subspace $\tilde{z} \in \mathcal{R}(D^T \otimes I_p)$, thus $\mathcal{R}(W)$ with

$$W = \begin{bmatrix} I & 0 \\ 0 & D^T \otimes I_p \end{bmatrix} \tag{47}$$

is an A-invariant subspace. Rather than investigate the dynamics of (12) directly, we instead consider the dynamics in this A-invariant space. To this end, note that $AW = WC$ with

$$C = \begin{bmatrix} -kI & -D_Q(D^T \otimes I_p) \\ I & 0 \end{bmatrix}. \tag{48}$$

Thus, we can consider the stability properties of $\dot{\eta} = C\eta$, keeping in mind that

$$\mathcal{N}(C) = \mathcal{N}(W) = \mathcal{R}\left(\begin{bmatrix} 0 & (\mathbf{1}^T \otimes I_p) \end{bmatrix}^T\right) \tag{49}$$

is nontrivial. In particular, (45) restricted to the subspace $\mathcal{R}(W)$ is asymptotically stable if and only if the equilibria subspace $\mathcal{N}(W)$ of $\dot{\eta} = C\eta$ is asymptotically stable. Via block matrix inversion formulae using Schur complements, we have that the characteristic polynomial of C is

$$\det(sI - C) = \det(s^2 I + ksI + D_Q(D^T \otimes I_p)) \tag{50}$$

$$= \prod_{i=1}^{mp} (s^2 + ks - \mu_i(-D_Q(D^T \otimes I_p))) \tag{51}$$

where $\mu_i(-D_Q(D^T \otimes I_p))$ is the ith eigenvalue of $-D_Q(D^T \otimes I_p)$. Thus each eigenvalue of $-D_Q(D^T \otimes I_p)$ generates two eigenvalues of C, specifically

$$\mu_{i^+,i^-} = \frac{-k \pm \sqrt{k^2 + 4\mu_i(-D_Q(D^T \otimes I_p))}}{2}. \tag{52}$$

In general, $\text{Re}[\mu_i] < 0$ does not imply $\text{Re}[\mu_{i^+,i^-}] < 0$, however this implication is true if

$$k > \sqrt{\text{Im}[\mu_i]^2 / (|\text{Re}[\mu_i]|)}. \tag{53}$$

By assumption, $-D_Q(D^T \otimes I_p)$ has p eigenvalues at 0 and the rest are in the open left half plane. Thus, if k satisfies (53) for all nonzero μ_i, then the p zero eigenvalues of $-D_Q(D^T \otimes I_p)$ will generate p zero eigenvalues of C corresponding to the equilibria subspace $\mathcal{N}(W)$, and the remaining eigenvalues will be in the open left half plane.

(only if) Via the above argument, it is clear that if $\text{Re}[\mu_i] \geq 0$, then this μ_i corresponds to an eigenvalue of (46) with real part greater than or equal to zero. Thus if (44) has additional unstable modes, this will correspond to unstable modes of (46). ∎

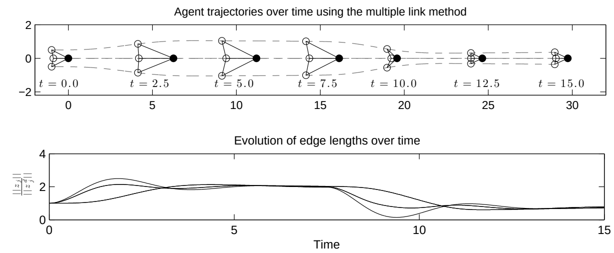
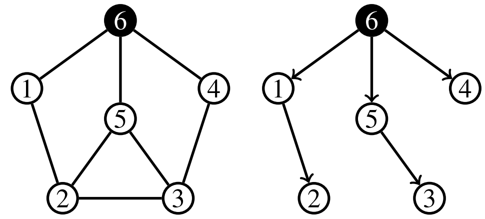


Fig. 2. Simulation results for a 4-agent, 5-edge formation using the multiple link method in which the agents are initialized in the correct formation such that $z(0) = z^d(0)$ and $\lambda = 2$ on $t \in [0, 7.5)$ and $\lambda = 0.75$ on $t \in [7.5, 15]$. The shaded node indicates the leader.



(a) (b)

Fig. 3. (a) Regular pentagon formation with (b) monitoring subgraph where agent 6 is the leader.

V. Simulation Results

Example 1. Dynamic desired formation scale.

While the results derived in this paper assume a static λ, it is clear that if λ does not change quickly or often with respect to the formation dynamics, then the desired scaling can be used to dynamically adjust the formation over time. In Figure 2, the formation is initialized so that $z(0) = z^d(0)$ with $\lambda(t) = \begin{cases} 2, & t \in [0, 7.5) \\ 0.75, & t \in (7.5, 15]. \end{cases}$

Example 2. Comparison of methods when condition (43) *is and is not satisfied.*

Consider the regular pentagon formation depicted in Fig. 3(a) with monitoring subgraph depicted in Fig. 3(b). This example formation does not satisfy (43). Indeed, Fig. 4 shows that, while both methods are stable, the multiple link method does not reach the desired scaled formation. The plot of $\frac{||z_j||}{||z_j^d||}$ shows that the lengths of the edges does not converge to a common value reflecting the fact that the subspace $\mathcal{R}(D^T \otimes I_p) \cap S$ has dimension larger than one. By adding edges between nodes 1 and 5 and nodes 4 and 5 to the desired formation, condition (43) is satisfied and (3)–(4) is achieved. Figure 5 shows that both methods converge to the desired formation in this case.

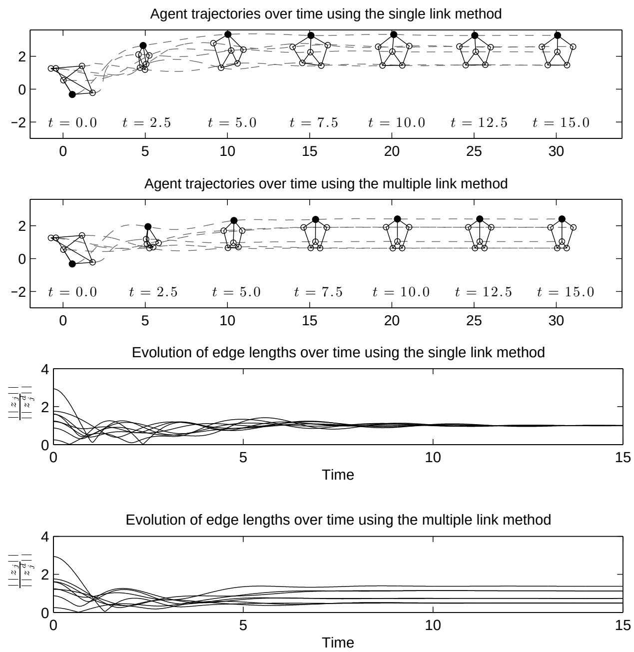


Fig. 4. Simulation results for a 6-agent, 8-edge formation that does not satisfy (43). The shaded node indicates the leader.

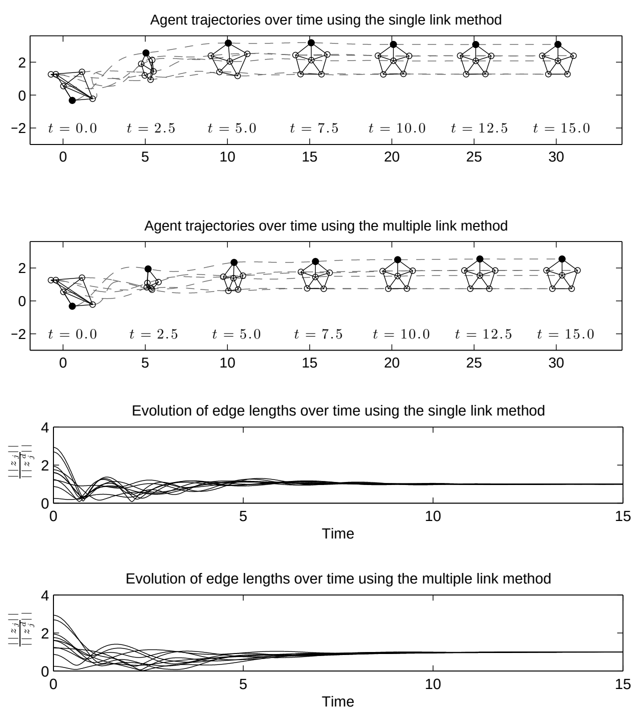


Fig. 5. Simulation results for a 6-agent, 10-edge formation that satisfies (43). The shaded node is the leader.

VI. Conclusions

In this paper, we present two distributed cooperative control strategies for formation control using only local relative position information of neighboring agents that allows a subset of the agents to determine the size of the formation. The ability to scale the formation size with only relative position information can be accomplished using only local sensors and is useful in environments where direct communication is not possible, yet flexibility in the achieved formation is desirable. For the single link method, we have presented an easily verified sufficient condition for stability derived via the small-gain theorem, and we have established a necessary and sufficient condition for stability in the case of the multiple link method.

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