

# Energy Management via Pricing in LQ Dynamic Games

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**Abstract**—This paper investigates the use of pricing mechanisms as a means to achieve a desired feedback control strategy among selfish agents in the context of HVAC resource allocation in buildings. We pose the problem of resource allocation as a linear-quadratic game with many dynamically coupled zone occupants (agents) and an uncoupled social planner. The social planner influences the game by choosing the quadratic dependence on control actions for each agent’s cost function. We propose a neighborhood-based simplification of the dynamic game that results in a more realistic and scalable framework than is considered in standard dynamic game theory. In addition, we construct the pricing design problem as a convex feasibility problem and apply our method to an eight zone building model.

## I. INTRODUCTION

Incentive theory is used to study problems with socio-economic considerations such as resource allocation among selfish agents. In such scenarios, the solution to which agents converge naturally is often inefficient from a societal point of view, and engineered systems exhibiting such behavior are appearing more frequently in the literature as technology becomes integrated into infrastructure [1]–[6]. It is important to accurately model these systems and develop control strategies that account for the interests of all the participating agents while meeting an organizational objective which may represent social welfare or common good.

Coordinating selfish agents to meet an organization objective through means such as prices, taxes, and rewards is broadly called *mechanism design* [7], and there has been considerable literature proposing the use of mechanisms as a means to achieve a socially optimal solution [8]–[12]. Related literature considers the design of local utility functions to achieve a socially optimal solution among cooperative agents with local information [13]–[15]. Pricing mechanisms allow for a social planner to shape the strategy of the agents and close the gap between the centralized, socially optimal cost and the decentralized cost arising from selfish individual behavior.

In this paper, we consider pricing schemes for energy management in buildings with multiple self-interested occupants. Energy consumption due to heating and cooling of buildings accounts for approximately 40% of all energy

usage in the U.S. [16]. Many control methods, such as model predictive control, have been proposed as a means to improve the efficiency of the heating and cooling process [17]–[19]. In addition, many economic solutions have been proposed to reduce consumption through economic incentives such as dynamic pricing and smart meter technology [20], [21].

We propose a combination of pricing mechanisms and control as a means to improve efficiency by reducing energy costs while maintaining a level of comfort in the building that allows for maximum productivity. We study the pricing problem in a dynamic setting where the zones are thermodynamically coupled. In addition, we propose a novel approach to the noncooperative game formulation that allows our results to scale well with the number of competing agents.

In Section II, we formally define the pricing mechanism design problem in the discrete time framework for energy management in buildings. In Section III, we state and prove our main result. In Section IV, we consider revenue neutral pricing mechanisms, and in Section V, we apply the theory developed to the problem of energy management for a physics-based model of a building. In Section VI, we make concluding remarks.

## II. PROBLEM FORMULATION

### A. Dynamical System

We consider a building with of  $p$  zones indexed  $1, \dots, p$ . We assume each zone has state  $x_i \in \mathbb{R}$  representing the temperature of zone  $i$  and that each zone consists of a selfish player that controls the input<sup>1</sup>  $u_i \in \mathbb{R}$  representing input air temperature. We denote vector transpose by  $'$  and write the stacked state and input vectors as

$$x \triangleq [x_1 \ \dots \ x_p]', \quad u \triangleq [u_1 \ \dots \ u_p]'$$

The system evolves under the following dynamics:

$$x[k+1] = A[k]x[k] + \sum_{i=1}^p B_i[k]u_i[k] + d[k] \quad (1)$$

for matrices  $A[k] \in \mathbb{R}^{p \times p}$ ,  $B_i[k] \in \mathbb{R}^{p \times 1}$ , and disturbance vector  $d[k] \in \mathbb{R}^p$  for all  $i$  and  $k = 1, \dots, T$ . The disturbance vector can include the effect of outside air temperature or unmodeled dynamics and is defined in further detail in the examples of Section V. We assume players have a desired zone temperature  $x_i^{\text{des}}[k] \in \mathbb{R}$  for all  $k$  and define  $x^{\text{des}}[k] \triangleq [x_1^{\text{des}}[k] \ \dots \ x_p^{\text{des}}[k]]'$ .

<sup>1</sup>The analysis follows exactly the same with mild notational changes if the zone state  $x_i$  or input  $u_i$  are multidimensional vectors representing other quantities.

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We consider a neighborhood  $\mathcal{N}_i \subset \{1, \dots, p\}$  of each player and define the Nash equilibrium in terms of a truncated system defined relative to this neighborhood. For example,  $\mathcal{N}_i$  may consist of zones that are physically adjacent to zone  $i$ . We assume  $i \in \mathcal{N}_i$ .

We now associate the following truncated dynamical system with each player. Let  $\mathcal{O}_i = \{1, \dots, p\} \setminus \mathcal{N}_i$ . After reordering players, we partition the players and inputs such that (1) can be written as

$$\begin{aligned} \begin{bmatrix} x_{\mathcal{N}_i}[k+1] \\ x_{\mathcal{O}_i}[k+1] \end{bmatrix} &= \begin{bmatrix} A_{\mathcal{N}_i}[k] & A_{\mathcal{N}_i, \mathcal{O}_i}[k] \\ A_{\mathcal{O}_i, \mathcal{N}_i}[k] & A_{\mathcal{O}_i}[k] \end{bmatrix} \begin{bmatrix} x_{\mathcal{N}_i}[k] \\ x_{\mathcal{O}_i}[k] \end{bmatrix} \\ &+ \sum_{j \in \mathcal{N}_i} \begin{bmatrix} B_{j, \mathcal{N}_i}[k] \\ B_{j, \mathcal{O}_i}[k] \end{bmatrix} u_j[k] \\ &+ \sum_{j \in \mathcal{O}_i} \begin{bmatrix} B_{j, \mathcal{N}_i}[k] \\ B_{j, \mathcal{O}_i}[k] \end{bmatrix} u_j[k] + \begin{bmatrix} d_{\mathcal{N}_i}[k] \\ d_{\mathcal{O}_i}[k] \end{bmatrix} \end{aligned}$$

where  $x_{\mathcal{N}_i}$  is a stacked column of states corresponding to the set of players  $\mathcal{N}_i$ , similarly for  $x_{\mathcal{O}_i}$  and the set  $\mathcal{O}_i$ ,  $\{A_{\mathcal{N}_i}[k], A_{\mathcal{N}_i, \mathcal{O}_i}[k], A_{\mathcal{O}_i, \mathcal{N}_i}[k], A_{\mathcal{O}_i}[k]\}$  is an appropriately formed partition of  $A[k]$ ,  $\{B_{j, \mathcal{N}_i}[k], B_{j, \mathcal{O}_i}[k]\}$  is an appropriately formed partition of  $B_j[k]$  for each  $j$ , and  $\{d_{\mathcal{N}_i}[k], d_{\mathcal{O}_i}[k]\}$  is an appropriately formed partition of  $d[k]$ .

To be more explicit, we index the elements of  $\mathcal{N}_i$  as  $\{l_1^i, \dots, l_{|\mathcal{N}_i|}^i\}$ . Without loss of generality, we assume  $l_1^i = i$ . Let  $e_j$  denote the  $j$ th standard basis vector and define

$$\Psi_{\mathcal{N}_i} \triangleq \begin{bmatrix} e'_{l_1^i} & \dots & e'_{l_{|\mathcal{N}_i|}^i} \end{bmatrix}',$$

and similarly define  $\Psi_{\mathcal{O}_i}$  for an indexing of  $\mathcal{O}_i$ . Thus  $x_{\mathcal{N}_i}[k] \triangleq \Psi_{\mathcal{N}_i} x[k]$ ,  $u_{\mathcal{N}_i}[k] \triangleq \Psi_{\mathcal{N}_i} u[k]$ ,  $A_{\mathcal{N}_i}[k] \triangleq \Psi_{\mathcal{N}_i} A[k] \Psi_{\mathcal{N}_i}'$ ,  $A_{\mathcal{N}_i, \mathcal{O}_i}[k] \triangleq \Psi_{\mathcal{N}_i} A[k] \Psi_{\mathcal{O}_i}'$ , etc.

From this partitioned system, we extract the truncated system

$$\begin{aligned} x_{\mathcal{N}_i}[k+1] &= A_{\mathcal{N}_i}[k] x_{\mathcal{N}_i}[k] + \sum_{j \in \mathcal{N}_i} B_{j, \mathcal{N}_i}[k] u_j[k] \\ &+ d_{\mathcal{N}_i}[k] + w_i[k] \end{aligned} \quad (2)$$

where  $w_i[k] = A_{\mathcal{N}_i, \mathcal{O}_i}[k] x_{\mathcal{O}_i}[k] + \sum_{j \in \mathcal{O}_i} B_{j, \mathcal{N}_i}[k] u_j[k]$ .

We assume agent  $i$  considers the dynamical system (2), but rather than modeling  $w_i$  explicitly, each agent assumes

$$w_i[k] = A_{\mathcal{N}_i, \mathcal{O}_i}[k] \bar{x}_{\mathcal{O}_i}[k] \quad (3)$$

where  $\bar{x}_{\mathcal{O}_i}[k]$  is agent  $i$ 's belief about the states of the other agents at time  $k$ . We assume

$$\bar{x}_{\mathcal{O}_i}[k] = x_{\mathcal{O}_i}^{\text{des}}[k] \triangleq \Psi_{\mathcal{O}_i} x^{\text{des}}[k], \quad (4)$$

although other choices exist and the theory extends to the case where agents assume a statistical distribution for  $w_i[k]$ .

Equations (3) and (4) are reasonable when distant zones have minimal thermal influence on a zone and its neighbors and when  $u_j[k]$  only directly affects  $x_j[k]$ . Similar truncations or clustering can be found elsewhere in the literature, e.g. [22]. Two important features of this truncation are:

- a more realistic assumption on the actions of each agent, as we explicitly describe the neighborhood  $\mathcal{N}_i$  used in agent  $i$ 's model (2) of the dynamical system (1),

- a scalable solution concept, as the size of neighbor sets will often be limited and relatively small, even for large buildings with many zones.

## B. Cost

We model the cost incurred by each agent  $i$  as a quadratic cost  $J_i$  given by

$$\begin{aligned} J_i &\triangleq q_i[T] (x_i[T] - x_i^{\text{des}}[T])^2 + \sum_{k=0}^{T-1} q_i[k] (x_i[k] - x_i^{\text{des}}[k])^2 \\ &+ (u_{\mathcal{N}_i}[k] - u_{\mathcal{N}_i}^{\text{free}}[k])' R^i[k] (u_{\mathcal{N}_i}[k] - u_{\mathcal{N}_i}^{\text{free}}[k]) \end{aligned} \quad (5)$$

where  $q_i[k] \in \mathbb{R}_{\geq 0}$  represent discomfort costs,  $R^i[k] \in \mathbb{R}^{|\mathcal{N}_i| \times |\mathcal{N}_i|}$ ,  $R^i[k] \succeq 0$  represent heating and cooling costs,  $u_{\mathcal{N}_i}^{\text{free}}[k] = \Psi_{\mathcal{N}_i} u^{\text{free}}[k] \in \mathbb{R}^{|\mathcal{N}_i|}$  where  $u^{\text{free}}[k] = [u_1^{\text{free}}[k]' \dots u_p^{\text{free}}[k]']'$  and  $u_j^{\text{free}}[k]$  is the ambient outdoor temperature.

## C. Nash Equilibrium of Player Strategies

We assume each player's strategy space is the class of affine, memoryless state feedback strategies to (2), denoted by  $\Gamma_i$ . We assume the zone occupants are rational and collectively play a Nash equilibrium. A *Nash equilibrium* is defined as a collection of zone control strategies  $(\gamma_1^*, \dots, \gamma_p^*)$  with  $\gamma_i^* \in \Gamma_i$  such that the following holds:

$$J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i, \gamma_{-i}^*) \quad \forall \gamma_i \in \Gamma_i, \quad \forall i$$

where  $\gamma_{-i}^*$  denotes the set of actions taken by players other than player  $i$ , i.e.  $-i \triangleq \{1, \dots, i-1, i+1, \dots, p\}$  and  $\gamma_{-i}^* \triangleq \{\gamma_j^*\}_{j \in -i}$ , and  $J_i(\gamma_i^*, \gamma_{-i}^*)$  indicates the cost (5) when strategies  $\gamma_i^*$  and  $\gamma_{-i}^*$  are used. By restricting to the class of memoryless state feedback control strategies, we ensure that the Nash equilibrium is unique when  $\mathcal{N}_i = \{1, \dots, p\}$ , see [23]. We conjecture uniqueness holds for general  $\mathcal{N}_i$  as well. Since  $\gamma_i$  is memory-less and affine, we always have  $\gamma_i = \{\gamma_i(\cdot, k)\}_{k=1}^T$  where

$$u_i[k] \triangleq \gamma_i(x_{\mathcal{N}_i}[k], k) = -K_i[k] x_{\mathcal{N}_i}[k] - \kappa_i[k]$$

for some collection of matrices  $K_i[k] \in \mathbb{R}^{1 \times |\mathcal{N}_i|}$  and scalars  $\kappa_i[k] \in \mathbb{R}$ . By an abuse of notation, we sometimes refer to  $u_i[k]$  or  $\{K_i[k], \kappa_i[k]\}_{k=1}^T$  as player  $i$ 's strategy.

## D. Introducing a Social Planner

We now introduce a social planner whose goal is to decrease the total cost incurred by the zone occupants and can measure  $u$  but does not know  $x^{\text{des}}$ . Thus the social planner influences the game by designing new cost matrices  $\hat{R}^i[k]$  such that the new cost incurred by each agent is

$$\begin{aligned} \hat{J}_i &\triangleq q_i[T] (x_i[T] - x_i^{\text{des}}[T])^2 + \sum_{k=0}^{T-1} q_i[k] (x_i[k] - x_i^{\text{des}}[k])^2 \\ &+ \sum_{k=0}^{T-1} (u_{\mathcal{N}_i}[k] - u_{\mathcal{N}_i}^{\text{free}}[k])' \hat{R}^i[k] (u_{\mathcal{N}_i}[k] - u_{\mathcal{N}_i}^{\text{free}}[k]). \end{aligned} \quad (6)$$

We refer to (5) as *nominal costs*, and refer to (6) as *costs under pricing scheme*  $\{\hat{R}^i[0], \dots, \hat{R}^i[T-1]\}_{i=1}^p$ , or simply *costs with pricing*. We refer to the Nash equilibrium defined in Section II-C as the Nash equilibrium under nominal costs, and similarly define a new Nash equilibrium under pricing using (6), and we denote this equilibrium by  $\{\hat{\gamma}_i^*\}_{i=1}^p$ .

**Problem Statement.** Design  $\hat{R}^i[k]$  for all  $i$  and  $k$  such that the resulting Nash Equilibrium under pricing is

$$u_i^*[k] = -K_i^{\text{des}}[k]x_{\mathcal{N}_i}[k] - \kappa_i^{\text{des}}[k] + u_i^{\text{free}}[k] \quad (7)$$

where  $K_i^{\text{des}}[k] \in \mathbb{R}^{1 \times |\mathcal{N}_i|}$  and  $\kappa_i^{\text{des}}[k] \in \mathbb{R}$  for all  $i$  and  $k$  are desired feedback gains established by the social planner to minimize  $J_L \triangleq \sum_{i=1}^p J_i$  (see Section III-A).

We let

$$\bar{K}_i^{\text{des}} \triangleq \begin{bmatrix} K_i^{\text{des}}[k] \\ \kappa_i^{\text{des}}[k] \end{bmatrix} \quad (8)$$

and  $\bar{x}_{\mathcal{N}_i}[k] \triangleq [x_{\mathcal{N}_i}[k]' \ 1]'$ , and we then write (7) as  $u_i^*[k] = -\bar{K}_i^{\text{des}}\bar{x}_{\mathcal{N}_i}[k] + u_i^{\text{free}}[k]$ .

### III. MAIN RESULTS

#### A. Social Planner's Desired Control

To minimize  $J_L = \sum_{i=1}^p J_i$ , the social planner considers (1), rewritten as

$$x[k+1] = A[k]x[k] + B[k]u[k] + d[k] \quad (10)$$

with  $B[k] = [B_1[k] \ \dots \ B_p[k]]$ . We now introduce the following augmented vectors:

$$\bar{x}[k] \triangleq [x[k]' \ 1]', \quad \bar{u}[k] \triangleq u[k] - u^{\text{free}}[k]. \quad (11)$$

We then have  $\bar{x}[k+1] = \bar{A}[k]\bar{x}[k] + \bar{B}[k]\bar{u}[k]$  where

$$\bar{A}[k] \triangleq \begin{bmatrix} A[k] & B[k]u^{\text{free}}[k] + d[k] \\ 0 & 1 \end{bmatrix}, \quad \bar{B}[k] \triangleq \begin{bmatrix} B[k] \\ 0 \end{bmatrix}. \quad (12)$$

Furthermore,

$$J_L = \bar{x}[T]'\bar{Q}[T]\bar{x}[T] + \sum_{k=0}^{T-1} (\bar{x}[k]'\bar{Q}[k]\bar{x}[k] + \bar{u}[k]R[k]\bar{u}[k]) \quad (13)$$

where

$$\begin{aligned} \bar{Q}[k] &\triangleq \begin{bmatrix} Q[k] & -Q[k]x^{\text{des}}[k] \\ -x^{\text{des}}[k]Q[k] & x^{\text{des}}[k]Q[k]x^{\text{des}}[k] \end{bmatrix} \\ Q[k] &\triangleq \text{diag}\{q_1[k], \dots, q_p[k]\} \\ R[k] &\triangleq \sum_i \Psi'_{\mathcal{N}_i} R^i[k] \Psi_{\mathcal{N}_i} \end{aligned} \quad (14)$$

and we assume  $R[k] \succ 0$ .

We solve the resulting optimal control problem using standard linear-quadratic regulator theory, summarized in the following lemma:

**Lemma 1.** Consider the dynamical system  $x[k+1] = A[k]x[k] + B[k]u[k]$  and assume  $R[k] \succ 0$  for  $k = 0, \dots, T-1$ ,  $Q[T] \succeq 0$ , and  $\begin{bmatrix} Q[k] & N[k] \\ N[k]' & R[k] \end{bmatrix} \succeq 0$  for  $k = 0, \dots, T-1$ . If there exists matrices  $P[k] \succeq 0$  such that  $P[T] = Q[T]$  and

$$\begin{aligned} P[k] &= Q[k] + A[k]'P[k+1]A[k] \\ &\quad - K[k]'(B[k]'P[k+1]B[k] + R[k])K[k] \quad \forall k \\ K[k] &\triangleq (B[k]'P[k+1]B[k] + R[k])^{-1} \\ &\quad \cdot (B[k]'P[k+1]A[k] + N[k]') \quad \forall k \end{aligned} \quad (15)$$

then the feedback law  $u[k] = -K[k]x[k]$  minimizes

$$J = x[T]'\bar{Q}[T]x[T] + \sum_{k=0}^{T-1} (x[k]'\bar{Q}[k]x[k] + u[k]R[k]u[k] + 2x[k]N[k]u[k]).$$

The minimizing cost is  $J^* = x[0]'P[0]x[0]$ .

*Proof:* See e.g. [24] for a proof of the special case when  $N[k] = 0$  for all  $k$ . The general case follows straightforwardly. ■

We use Lemma 1 to minimize (13) subject to the dynamics (10) using the augmentation technique (11)–(14). The resulting optimal controller is obtained via (15), and we denote this controller by  $\{\bar{K}[k]\}_{k=0}^{T-1}$ . We can partition each  $\bar{K}[k]$  along rows and obtain the collection  $\{\bar{K}_i[k]\}_{k=0}^{T-1}$  where  $\bar{K} = [\bar{K}_1[k]' \ \dots \ \bar{K}_p[k]']'$  and  $u_i[k] = -\bar{K}_i[k]\bar{x}[k] + u_i^{\text{free}}[k]$  is the team optimal control policy for (1) with total cost (13).

From the formulation given in Section II and equation (7) from the Problem Statement, each agent uses feedback of the truncated vector  $x_{\mathcal{N}_i}$  rather than  $x$ . This amounts to a sparsity constraint on the feedback gain matrix  $\bar{K}[k]$  calculated by the social planner. In this case, Lemma 1 does not apply. Optimal control with such sparsity constraints is an active area of research, see e.g. [25], [26]. However, we assume the social planner has a technique for either computing this optimal sparse controller, or a method for obtaining some other, near-optimal desired sparse controller. For example, the social planner could simply truncate the controllers  $\bar{K}_i[k]$  described above. Empirically, this is often near optimal, and this is the method used in the example of Section V. We denote the desired set of controllers as

$$u_i^{\text{des}}[k] = -\bar{K}_i^{\text{des}}[k]\bar{x}_{\mathcal{N}_i}[k] + u_i^{\text{free}}[k] \quad (16)$$

where we partition  $\bar{K}_i^{\text{des}}$  as in (8). We also define

$$\bar{K}^{\text{des}}[k] \triangleq [(\Psi'_{\mathcal{N}_1} \bar{K}_1^{\text{des}}[k] \Psi_{\mathcal{N}_1})' \ \dots \ (\Psi'_{\mathcal{N}_p} \bar{K}_p^{\text{des}}[k] \Psi_{\mathcal{N}_p})']'$$

so that  $u^{\text{des}}[k] = -\bar{K}^{\text{des}}[k]\bar{x}[k] + u^{\text{free}}[k]$ .

#### B. Designing Prices

We now consider the design of prices  $\{\hat{R}[k]\}_{i=1}^p$  for all  $k$  such that the social planner's desired set of controllers (16) constitutes a Nash equilibrium under pricing as described in section II-C. To this end, we assume  $u_j[k] = u_j^{\text{des}}[k]$  for all  $j \in -i$  and, substituting in (2),

$$\bar{x}_{\mathcal{N}_i}[k+1] = \bar{A}_{\mathcal{N}_i}[k]\bar{x}_{\mathcal{N}_i}[k] + B_{\mathcal{N}_i}[k]\bar{u}_i[k] + \bar{w}_i[k]$$

where  $\bar{A}_{\mathcal{N}_i}[k]$  is defined in (17) on the following page,

$$\begin{aligned} \bar{B}_{\mathcal{N}_i}[k] &\triangleq \begin{bmatrix} B_{\mathcal{N}_i}[k] \\ 0 \end{bmatrix}, \quad \bar{w}_i[k] \triangleq \begin{bmatrix} w_i[k] - A_{\mathcal{N}_i, \mathcal{O}_i}[k]x_{\mathcal{O}_i}[k] \\ 0 \end{bmatrix} \\ \bar{u}_i[k] &\triangleq u_i[k] - u_i^{\text{free}}[k], \end{aligned}$$

we assume (3) and (4), and

$$(B_{\mathcal{N}_j}[k]\bar{K}_j^{\text{des}}[k])_{\mathcal{N}_i} = \Psi_{\mathcal{N}_i}\Psi'_{\mathcal{N}_j}(B_{\mathcal{N}_j}[k]\bar{K}_j^{\text{des}}[k])\Psi_{\mathcal{N}_j}\Psi'_{\mathcal{N}_i},$$

i.e.  $(B_{\mathcal{N}_j}[k]\bar{K}_j^{\text{des}}[k])_{\mathcal{N}_i}$  is the matrix  $(B_{\mathcal{N}_j}[k]\bar{K}_j^{\text{des}}[k])$  re-ordered and truncated to match  $A_{\mathcal{N}_i}[k]$ .

Let  $\hat{R}_{s,t}^i$  denote the  $(s, t)$ th entry of  $\hat{R}^i$ . With  $u_j[k] = u_j^{\text{des}}[k]$  for  $j \in -i$ , we rewrite the cost under pricing (6) as

$$\begin{aligned} \hat{J}_i &= \bar{x}_{\mathcal{N}_i}[T]'\bar{Q}^i[T]\bar{x}_{\mathcal{N}_i}[T] + \sum_{k=0}^{T-1} \bar{x}_{\mathcal{N}_i}[k]'\bar{Q}^i[k]\bar{x}_{\mathcal{N}_i}[k] \\ &\quad + \sum_{k=0}^{T-1} \bar{u}_i[k]'\hat{R}_{i,i}^i[k]\bar{u}_i[k] + 2\bar{x}_{\mathcal{N}_i}[k]'\tilde{N}^i[k]\bar{u}_i \end{aligned}$$

$$\bar{A}_{\mathcal{N}_i}[k] \triangleq \begin{bmatrix} A_{\mathcal{N}_i}[k] - \sum_{j \in (\mathcal{N}_i \setminus \{i\})} (B_{\mathcal{N}_j}[k] \bar{K}_j^{\text{des}}[k])_{\mathcal{N}_i} & \sum_{j \in \mathcal{O}_i} (B_{\mathcal{N}_j}[k] u_{\mathcal{N}_j}^{\text{free}}[k])_{\mathcal{N}_i} + A_{\mathcal{N}_i, \mathcal{O}_i}[k] x_{\mathcal{O}_i}^{\text{des}}[k] + d_{\mathcal{N}_i}[k] \\ 0 & 1 \end{bmatrix} \quad (17)$$

$$\tilde{Q}^i[k] \triangleq \begin{bmatrix} Q_i & -q_i[k] x_i^{\text{des}}[k] \\ -q_i[k] x_i^{\text{des}}[k] & q_i[k] (x_i^{\text{des}}[k])^2 \end{bmatrix} + \Psi_{\mathcal{N}_i} \left( \sum_{s \neq i, t \neq i} \Psi'_{\mathcal{N}_s} (\bar{K}_s^{\text{des}}[k])' \hat{R}_{s,t}^i[k] \bar{K}_i^{\text{des}}[k] \right) \Psi'_{\mathcal{N}_i} \quad (18)$$

where  $\tilde{N}^i[k] \triangleq \Psi_{\mathcal{N}_i} \sum_{s \neq i} \Psi'_{\mathcal{N}_s} \bar{K}_s^{\text{des}}[k] \hat{R}_{s,i}^i[k]$ ,  $Q_i \triangleq \text{diag}\{q_i, 0, \dots, 0\}$ , and  $\tilde{Q}^i[k]$  is defined in (18).

**Theorem 1.** *If matrices  $\{\hat{R}^i[0], \dots, \hat{R}^i[T-1]\}_{i=1}^p$  and  $\{P_i[1], \dots, P_i[T]\}_{i=1}^p$  exists such that the convex feasibility problem below is feasible for all  $i \in \{1, \dots, p\}$ , then  $\{u_i^*[k] = u_i^{\text{des}}[k] = -\bar{K}_i^{\text{des}}[k] \bar{x}_{\mathcal{N}_i}[k] + u_i^{\text{free}}[k]\}_{i=1}^p$  is a Nash equilibrium of the game, thereby achieving the social planner's goal:*

$$\begin{aligned} P_i[T] &= \tilde{Q}^i[T], \quad \hat{R}_{ii}^i[k] \succ 0, \quad \begin{bmatrix} \tilde{Q}^i[k] & \tilde{N}^i[k] \\ (\tilde{N}^i[k])' & R_{ii}^i[k] \end{bmatrix} \succeq 0, \\ P_i[k] &= \tilde{Q}^i[k] + \bar{A}_{\mathcal{N}_i}[k]' P_i[k+1] \bar{A}_{\mathcal{N}_i}[k] \\ &\quad - (\bar{K}_i^{\text{des}}[k])' (\bar{B}_i[k]' P_i[k+1] \bar{B}_i[k] + \hat{R}_{ii}^i[k]) \bar{K}_i^{\text{des}}[k], \\ (\bar{B}_i[k]' P_i[k] \bar{B}_i[k] + \hat{R}_{ii}^i[k]) \bar{K}_i^{\text{des}}[k] &= \\ &\quad (\bar{B}_i[k]' P_i[k] \bar{A}_{\mathcal{N}_i}[k] + (\tilde{N}^i[k])'). \end{aligned}$$

*Proof:* This follows from applying Lemma 1 to the system derived above and is a modification of Theorem 3 from [27]. ■

#### IV. REVENUE NEUTRAL PRICING AS AN OBJECTIVE

The nominal costs often represent actual costs incurred by the actions of the agents such as energy costs, loss of productivity due to discomfort, *etc.* The social planner imposes new costs with pricing, but often must ultimately pay these nominal costs, thus it is desirable to minimize the difference between the costs imposed/collected by the social planner and the nominal costs.

Let  $\bar{x}[k+1] = \bar{A}^{\text{CL}}[k] \bar{x}[k]$  denote the closed loop dynamics of (12) where

$$\bar{A}^{\text{CL}}[k] \bar{x}[k] \triangleq \bar{A}[k] \bar{x}[k] + \bar{B}[k] \bar{u}[k]$$

with  $\bar{u}_i = \bar{K}_i^{\text{des}}[k] \bar{x}_{\mathcal{N}_i}[k]$ . Let  $\Phi[k] = \prod_{l=1}^k \bar{A}^{\text{CL}}[l-1]$  for  $k = 1, \dots, T-1$ ,  $\Phi[0] \triangleq I_p$ . Thus  $\bar{x}[k] = \Phi[k] \bar{x}[0]$ . It then follows from a straightforward induction argument that

$$\sum_i \hat{J}_i = x[0]' \left( \Phi[T]' \bar{Q}[T] \Phi[T] + \sum_{k=0}^{T-1} \Phi[k]' S[k] \Phi[k] \right) x[0] \quad (19)$$

with  $S[k] = \bar{Q}[k] + \bar{K}^{\text{des}}[k]' \left( \sum_{i=1}^p \Psi'_{\mathcal{N}_i} \hat{R}^i[k] \Psi_{\mathcal{N}_i} \right) \bar{K}^{\text{des}}[k]$ .

Let  $C$  be the nominal cost to the leader when players use the desired control. It is clear that  $\sum_i \hat{J}_i$  defined in (19) is an affine function of  $\{\hat{R}^i[0], \dots, \hat{R}^i[T-1]\}_{i=1}^p$ , and thus we can add

$$\min(C - |\sum_i \hat{J}_i|)$$

as an objective to the convex feasibility problem of Theorem 1.

#### V. EXAMPLE: PRICING FOR ENERGY MANAGEMENT IN BUILDINGS

We consider a physics-based thermodynamic model of a building and apply the theory for pricing developed in the previous sections.

##### A. Physics-Based Building Model Description

If we consider transient conduction and convection as well as the flow of air injected into each zone by the HVAC system, then the temperature in zone  $i$ , denoted  $T_i$ , evolves according to the following dynamics:

$$\rho v_i C_p \frac{dT_i}{dt} = \sum_{j \in N_i} h_{i,j} a_{i,j} (T_j - T_i) + h_{i,o} a_{i,o} (T_\infty - T_i) + \dot{m}_i C_p (T_i^s - T_i)$$

where  $v_i$  is the volume of air in the  $i$ -th zone,  $a_{i,j}$  is the area of the wall between zone  $j$  and  $i$ ,  $a_{i,o}$  is the total area of the exterior walls and roof of zone  $i$ ,  $h_{i,j}$  and  $h_{i,o}$  are the heat transfer coefficients of the wall between zone  $j$  and  $i$  and the heat transfer coefficient of the exterior walls (determined by the material properties) respectively,  $\dot{m}_i$  is the mass flow rate of air into zone  $i$ ,  $T_i^s$  is the supply air temperature for zone  $i$ ,  $T_\infty$  is the outside ambient air temperature,  $C_p$  is the specific heat of air and  $\rho$  is the density of air (see Table I for a list of parameter values).

Define  $x = [x_1 \dots x_p]'$   $\triangleq [T_1 \dots T_p]'$  to be the state vector of zone temperatures and  $u = [u_1 \dots u_p]'$   $\triangleq [T_1^s \dots T_p^s]'$  to be the control input, *i.e.* vector of supply air temperatures. Then the system dynamics are

$$\dot{x} = Ax + \sum_{i \in N_z} B_i u_i + d$$

where  $d_i = \frac{h_{i,o} a_{i,o}}{\rho v_i C_p} T_\infty$ , and  $A$  and  $B$  are defined entrywise:

$$A_{ij} = \begin{cases} \sum_{j \in N_i} \frac{h_{i,j} a_{i,j}}{\rho v_i C_p} + \frac{\dot{m}_i}{\rho v_i} + \frac{h_{i,o} a_{i,o}}{\rho v_i C_p}, & \text{if } i = j \\ \frac{h_{i,j} a_{i,j}}{\rho v_i C_p}, & \text{if } j \in N_i \\ 0, & \text{otherwise} \end{cases}$$

$$B_i = \begin{cases} \frac{\dot{m}_i}{\rho v_i}, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

The dynamics of the building are such that  $A$  is stable. We use an exact discretization of the physics-based model with a time step of one minute to obtain a discrete-time model.

##### B. Results

We solve the pricing design problem for the physics-based model using code written in MATLAB that employs YALMIP [29] to solve the optimization problem. The cost

TABLE I

PARAMETERS FOR NUMERICAL SIMULATIONS<sup>1</sup>

Parameter	Value	Parameter	Value
$\dot{V}^2$	0.25 m <sup>3</sup> ·s <sup>-1</sup>	$\rho$	1.18 kg·m <sup>-3</sup>
$C_p$	1000 W·s·kg <sup>-1</sup> ·K <sup>-1</sup>	$h_{i,o}$	8.9W·m <sup>-2</sup> ·K <sup>-1</sup>
$T_\infty$	18°C	$\dot{m}$	0.295 kg·s <sup>-1</sup>

<sup>1</sup> Values from [28].<sup>2</sup>  $\dot{V}$  is the volumetric flow and the value corresponds to 500 CFM.

functions are composed of comfort costs and energy costs where we base comfort costs on the productivity relative to comfort level [30], detailed below.

Considering the nominal costs (5), we assume  $q_i[k] = q$  for all  $i, k$  for some  $q$  and assume  $R^i[k] = \text{diag}\{r, 0, \dots, 0\}$  for all  $i, k$  for some  $r$ . Thus we see that the cost savings depends only on the ratio of  $q$  to  $r$ , which we call the  $Q : R$  ratio. To develop a realistic ratio, we estimate the cost of energy to be \$0.13/kWh [31], and we use  $Q_{\text{sys}} = \dot{m}C_p(\Delta T)$  to calculate the amount of energy required to increase a zone's temperature by 1°C. To determine the comfort cost  $q$ , we note that, empirically, the ratio of total employee cost to building cost is roughly 13 to 1 [32], employees are roughly valued at \$25–\$30 per hour [33], and that there is a decrease of 2% in productivity per 1°C temperature increase [34], [35]. Using these values, the  $Q : R$  ratio is approximately 3 : 1. Motivated by this calculation, we investigate  $Q : R$  ratios ranging from 1 : 1 to 10 : 1.

In addition, we remark that resource allocation in large buildings can, in practice, be done very inefficiently due to the noncooperative nature of the local zone controllers. It is common for local controllers (thermostats) to use proportional control schemes where the gains used are determined based on heuristics. Hence, we compare the results under the pricing scheme to a naive proportional controller scheme. In the numerical examples presented, the naive proportional controller scheme we use is formed by taking the diagonal entries of the social planner's desired controller. We compute the Nash equilibrium under the nominal costs using the method of Lyapunov iterations [36]. We refer to this Nash equilibrium as the *nominal Nash equilibrium* and the resulting control law as the *nominal Nash controller*. We compute the percent savings gained by using the pricing scheme versus both the cost under the nominal Nash controller and the cost under the proportional controller.

The floor plan for the numerical examples is provided in Fig. 1, and the building height is assumed to be 5m, from which values of  $a_{i,j}$  and  $a_{i,o}$  are inferred. In Table II we report the percent savings gained by implementing the pricing scheme  $\{\hat{R}^i[0], \dots, \hat{R}^i[T-1]\}_{i=1}^p$  versus the naive proportional controller and nominal Nash controller. In the numerical example we allow the interior walls to be 4" of concrete block with rectangular cores. As the ratio of productivity cost to energy cost increases, the savings under pricing as compared to the proportional controller increases. The savings compared to the nominal Nash solution is maximum at the 5 : 1 ratio.

In Table III, we report the savings using pricing versus the

TABLE II

PERCENT SAVINGS FOR VARIOUS  $Q : R$  RATIOS<sup>1</sup>

$Q : R$ Ratio	Proportional vs. Pricing <sup>2</sup>	Nominal Nash vs. Pricing <sup>2</sup>
1 : 1	5.49%	2.33%
2 : 1	12.0%	3.41%
3 : 1	18.3%	3.91%
4 : 1	24.1%	4.13%
5 : 1	29.3%	4.20%
6 : 1	33.7%	4.18%
7 : 1	37.7%	4.11%
8 : 1	41.1%	4.02%
9 : 1	44.1%	3.91%
10 : 1	46.7%	3.79%

<sup>1</sup> Interior walls modeled as 4" concrete with heat transfer coefficient  $h_{i,j} = 10.89$  [W/(m<sup>2</sup>K)].<sup>2</sup> We compare the cost under a naive proportional controller and the nominal Nash cost to the cost with pricing over an eight hour day.

TABLE III

PERCENT SAVINGS FOR THREE TYPES OF INTERIOR WALLS<sup>1</sup>

Heat Transfer Coefficient (Values from [28])	Proportional vs. Pricing <sup>2</sup>	Nominal Nash vs. Pricing <sup>2</sup>
Steel studded wall, batting, 1/4" Gypsum Board $h_{i,j} = 5.4$ [W/(m <sup>2</sup> K)]	11.9%	1.52%
4" concrete block, rectangular core, $h_{i,j} = 10.89$ [W/(m <sup>2</sup> K)]	29.3%	4.20%
Air (at atmosphere) $h_{i,j} = 24$ [W/(m <sup>2</sup> K)]	52.3%	9.15%

<sup>1</sup> We assume  $Q : R = 5 : 1$ <sup>2</sup> We compare the cost under a naive proportional controller and the nominal Nash cost to the cost with pricing over an eight hour day.

proportional controller as well as the nominal Nash controller when the  $Q : R$  ratio is fixed at 5 : 1 and we vary the heat transfer coefficient over three values corresponding to steel studded walls with batting and 1/4" Gypsum board (standard insulation), 4" concrete block with rectangular cores (poor insulation), and air at atmospheric conditions (no insulation), which is motivated by the common practice in buildings with large open areas such as laboratories where the space is artificially divided into multiple zones with separate thermostats. Based on these results, we conjecture that, especially for poorly insulated buildings, the pricing scheme we present could provide considerable energy savings.

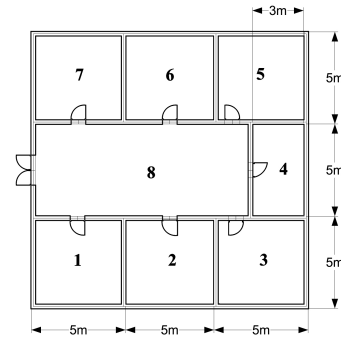


Fig. 1. Floor Plan for Building Example

In all of the numerical examples, the solutions are revenue neutral as described in Section IV. When the solution is revenue neutral and the pricing scheme is implemented, the

total nominal cost incurred when players use the desired strategy is redistributed across the players to prevent unilateral deviations from the desired control.

## VI. CONCLUSION

We provide a framework for designing prices in a building energy management setting to incentivize socially desirable control policies among the building occupants. We view the building as a dynamical system with selfish occupants and provide a novel approach whereby each agent considers a simplified version of the system dynamics. This reduction enables our results to be scalable and more realistic. We define the Nash equilibrium in terms of this simplification and show that designing prices to achieve a social planner-specified control strategy is a convex feasibility problem. We demonstrate our technique on an eight zone, physics-based building model and show that our technique can conservatively save 3–4% in costs. Our results also show that, depending on the nominal control strategies employed, savings can potentially be much higher.

Future directions of research include considering switched linear systems to model heating/cooling modes or linearized operating points. We are also working to apply our results to a building on the UC Berkeley campus.

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