Dynamical Properties of a Compartmental Model for Traffic Networks

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***Abstract*— In this paper, we propose a traffic network flow model particularly suitable for qualitative analysis as a dynamical system. Flows at a junction are determined by downstream *supply* of capacity (lack of congestion) as well as upstream *demand* of traffic wishing to flow through the junction. This approach is rooted in the celebrated Cell Transmission Model for freeway traffic flow, and we analyze resulting equilibrium flows and convergence properties.**

I. Introduction

Compartmental systems are a broad modeling paradigm to study fluid-like flow of a single substance among interconnected "compartments" [1], [2]. The main contribution of this paper is to propose and analyze a compartmental model of freeway traffic networks that is amenable to analysis as a dynamical system. Existing approaches are often well-suited for simulations or for validation/fitting with empirical data, but the available literature often gives little insight into the network-level, qualitative properties of the dynamics. For example, models such as [3], [4] and the celebrated *Cell Transmission Model (CTM)* [5], [6] were primarily developed for simulation with few analytical results available. The primary exception is [7] which provides a thorough investigation of the CTM when modeling a stretch of highway with onramp queues but does not consider more general networks.

We propose a model that encompasses the CTM and extends the model to general nonlinear supply and demand functions and to more general network topologies. In our proposed model, we consider a traffic network composed of road *links* interconnected at *junctions*. In keeping with the philosophy of the CTM, the flow of traffic through a junction is determined by the available *supply* of downstream road space into which vehicles can flow and upstream *demand* of vehicles wishing to flow into a given link.

Our work is related to the *dynamical flow networks* recently proposed in [8], [9] and further studied in [10]. In [8], [9], downstream supply is not considered and therefore the flow exiting a link is equal to the link's demand. Thus downstream congestion does not affect upstream flow, an unrealistic assumption for traffic modeling. In [10], the authors allow flow to depend on the density of downstream links, but the paper focuses on throughput optimality of a particular class of routing policies that does not accomodate most models of traffic flow, including the *proportional-priority, first-in-first-out* rule considered in this paper, and limited theoretical results are given for general routing policies.

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This paper is organized as follows: In Section II, we propose the traffic network model. In Section III, we assume constant input demand at onramps and provide a full characterization of the existence and uniqueness of equilibrium flows, even when onramp demands exceed network capacity. We then investigate convergence properties when the onramp demands are less than capacity and, for a specific class of networks, when they are greater than capacity. Due to space limitations, we exclude some proofs.

II. Dynamic Model of Traffic

A. Network structure

A traffic network consists of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{O})$ with *junctions* $\mathcal{V}$ and *ordinary links* $\mathcal{O}$ along with a set of *onramps* $\mathcal{R}$ which serve as entry points into the network. For $l \in \mathcal{O}$, let $\sigma(l)$ denote the head vertex of link l and let $\tau(l)$ denote the tail vertex of link l, and traffic flows from $\tau(l)$ to $\sigma(l)$. Each onramp $l \in \mathcal{R}$ directs an exogenous input flow onto $\mathcal{G}$ via a junction, and $\sigma(l) \in \mathcal{V}$ for $l \in \mathcal{R}$ denotes the entry junction for onramp l. By convention, $\tau(l) = \emptyset$ for all $l \in \mathcal{R}$.

Assumption 1. *The traffic network graph $\mathcal{G}$ is acyclic*[1].

Let $\mathcal{L} \triangleq \mathcal{O} \cup \mathcal{R}$. For each $v \in \mathcal{V}$, we denote by $\mathcal{L}_v^{\text{in}} \subset \mathcal{L}$ the set of incoming links to node v and by $\mathcal{L}_v^{\text{out}} \subset \mathcal{L}$ the set of outgoing links, *i.e.* $\mathcal{L}_v^{\text{in}} = \{l : \sigma(l) = v\}$ and $\mathcal{L}_v^{\text{out}} = \{l : \tau(l) = v\}$. We assume $\mathcal{L}_v^{\text{in}} \neq \emptyset$ for all $v \in \mathcal{V}$, thus the network flows start at onramps. Furthermore, we assume $\mathcal{L}_{\sigma(l)}^{\text{out}} \neq \emptyset$ for all $l \in \mathcal{R}$, *i.e.* onramps always flow into at least one ordinary link downstream. We define

$$\mathcal{R}^{\text{start}} \triangleq \{l \in \mathcal{R} : \mathcal{L}_{\sigma(l)}^{\text{in}} \cap \mathcal{O} = \emptyset\} \tag{1}$$

to be the set of links that lead to junctions that have only onramps as incoming links, and

$$\mathcal{V}^{\text{sink}} \triangleq \{v \in \mathcal{V} : \mathcal{L}_v^{\text{out}} = \emptyset\} \tag{2}$$

to be the set of junctions that have no outgoing links. An example network is show in Fig. 1.

B. Link supply and demand

For each link $l \in \mathcal{O}$, we associate the time-varying density $\rho_l(t) \in [0, \rho_l^{\text{jam}}]$ where $\rho_l^{\text{jam}} \in (0, \infty)$ is the *jam density* of link l. For $l \in \mathcal{R}$, we associate the time-varying density

[1] Acyclicity is a reasonable assumption when modeling a portion of the road network of particular interest. For example, the road network leading out of a metropolitan area during the evening commute may be modeled as an acyclic graph where road links leading towards the metropolitan area are not modeled due to low utilization by commuters.

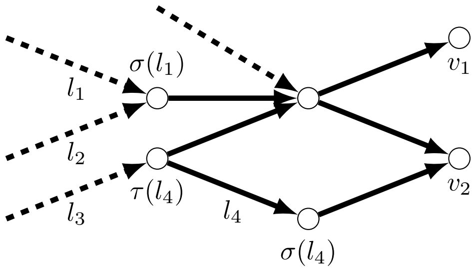
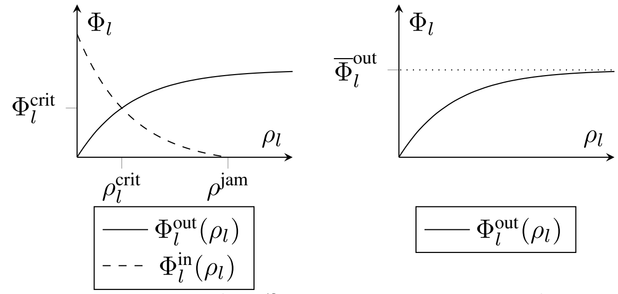


Fig. 1. An example traffic network. Ordinary links are indicated by a solid arrow, onramps are indicated by a dashed arrow. In the figure, $\mathcal{R}^{\text{start}} = \{l_1, l_2, l_3\}$ and $\mathcal{V}^{\text{sink}} = \{v_1, v_2\}$.



(a) Ordinary link, $l \in \mathcal{O}$ (b) Onramp link, $l \in \mathcal{R}$

Fig. 2. Plot of prototypical supply and demand functions $\Phi^{\text{in}}(\rho)$ and $\Phi^{\text{out}}(\rho)$ for (a) an ordinary road link, and prototypical demand function $\Phi^{\text{out}}(\rho)$ for (b) an onramp link. For onramp $l \in \mathcal{R}$, the input flow is assumed exogenously prescribed by $d_l(t)$.

$\rho_l(t) \in [0, \infty)$, thus onramps have no maximum density. We define $\rho \triangleq \{\rho_l\}_{l \in \mathcal{L}}$.

Furthermore, we assume each $l \in \mathcal{L}$ possesses a *demand* function $\Phi_l^{\text{out}}(\rho_l)$ that quantifies the amount of traffic on that link wishing to flow downstream, and we assume each $l \in \mathcal{O}$ possesses a *supply* function $\Phi_l^{\text{in}}(\rho_l)$. We make the following assumption on the supply and demand functions:

Assumption 2. *For each $l \in \mathcal{O}$:*

- *The demand function $\Phi_l^{out}(\rho_l) : [0, \rho_l^{jam}] \to \mathbb{R}_{\geq 0}$ is strictly increasing and continuously differentiable on $(0, \rho_l^{jam})$ with $\Phi_l^{out}(0) = 0$, and $\frac{d}{d\rho_l}\Phi_l^{out}(\rho_l)$ is bounded above.*
- *The supply function $\Phi_l^{in}(\rho_l) : [0, \rho_l^{jam}] \to \mathbb{R}_{\geq 0}$ is strictly decreasing and continuously differentiable on $(0, \rho_l^{jam})$ with $\Phi_l^{in}(\rho_l^{jam}) = 0$, and $\frac{d}{d\rho_l}\Phi_l^{in}(\rho_l)$ is bounded below.*

For each $l \in \mathcal{R}$:

- *The demand function $\Phi_l^{out}(\rho_l) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is increasing and continuously differentiable on $(0, \infty)$ with $\Phi_l^{out}(0) = 0$, and $\Phi_l^{out}(\rho_l)$ is bounded above with supremum $\overline{\Phi}_l^{out} \triangleq \sup \Phi_l^{out}(\rho_l)$. Furthermore, there exists[2] $M_l > 0$ such that $\frac{d}{d\rho_l}\Phi_l^{out}(\rho_l) \leq M_l(1+\rho_l)^{-2}$.*

Assumption 2 implies that for each $l \in \mathcal{O}$, there exists unique ρ_l^{crit} such that $\Phi_l^{\text{out}}(\rho_l^{\text{crit}}) = \Phi_l^{\text{in}}(\rho_l^{\text{crit}}) =: \Phi_l^{\text{crit}}$.

Remark 1. *The assumption of differentiability and strict monotonicity simplifies the exposition below but can be relaxed. For example, in some examples below we consider supply and demand functions that are piecewise differentiable.*

Fig. 2 depicts examples of supply and demand functions for ordinary links and onramp links.

C. Dynamic Model

We now describe the time evolution of the densities on each link. The domain of interest is

$$\mathcal{D} \triangleq \{\rho : \rho_l \in [0, \infty) \ \forall l \in \mathcal{R} \text{ and } \rho_l \in [0, \rho_l^{\text{jam}}] \ \forall l \in \mathcal{O}\}, \tag{3}$$

which is shown to be positively invariant for the model described in the sequel as stated in Proposition 1. Let $\mathcal{D}^\circ$ denote the interior of $\mathcal{D}$.

For each onramp $l \in \mathcal{R}$, we assume there exists exogenous *input flow* $d_l(t)$ that enters the link. Furthermore, for each $l \in \mathcal{R}$ there exists an output flow function $f_l^{\text{out}}(\rho)$, and for $l \in \mathcal{O}$ there exists input and output flow functions $f_l^{\text{in}}(\rho)$ and $f_l^{\text{out}}(\rho)$, respectively, such that

$$\dot{\rho}_l = F_l(\rho, t) \triangleq \begin{cases} d_l(t) - f_l^{\text{out}}(\rho) & \text{if } l \in \mathcal{R} \\ f_l^{\text{in}}(\rho) - f_l^{\text{out}}(\rho) & \text{if } l \in \mathcal{O} \end{cases} \tag{4}$$

where the functions $f_l^{\text{in}}(\rho)$ and $f_l^{\text{out}}(\rho)$ are defined according to the rule defined below. When $d_l(t) \equiv d_l$ for constant d_l for all $l \in \mathcal{R}$, the dynamics are autonomous and we write $F_l(\rho)$ instead. We define $F(\rho, t) \triangleq \begin{bmatrix} F_1(\rho, t) & \cdots F_{|\mathcal{L}|}(\rho, t) \end{bmatrix}'$ for some enumeration of $|\mathcal{L}|$ where $'$ denotes transpose, and we similarly define $F(\rho)$ when the dynamics are autonomous.

We assume that for each $v \in \mathcal{V}$ there exist *split* ratios

$$\beta_{lk}^v \in [0,1] \quad \forall l \in \mathcal{L}_v^{\text{in}} \quad \forall k \in \mathcal{L}_v^{\text{out}} \tag{5}$$

describing the fraction of vehicles flowing out of link l that are routed to link k. We note that $\sum_{k \in \mathcal{L}_v^{\text{out}}} \beta_{lk}^v \leq 1$ for all $l \in \mathcal{L}_v^{\text{in}}$ and define

$$\gamma_l \triangleq 1 - \sum_{k \in \mathcal{L}_{\sigma(l)}^{\text{out}}} \beta_{lk}^{\sigma(l)} \tag{6}$$

to be the fraction of the outflow on link l that is routed *off* the network via, *e.g.*, an infinite capacity offramp.

To ensure continuity of $f_l^{\text{out}}(\cdot)$ defined below, we make the following assumption:

Assumption 3. *If $v \notin \mathcal{V}^{sink}$, then $\beta_{lk}^v > 0$ for all $l \in \mathcal{L}_v^{in}$ and all $k \in \mathcal{L}_v^{out}$.*

A large variety of phenomenological rules for determining the outflows of road links have been proposed in the literature; see [12], [13] for several examples. We employ the *proportional priority, first-in-first-out(PP/FIFO)* rule for junctions adapted from [14]:

[2]The bound on the derivative of $\Phi_l^{\text{out}}(\rho_l)$ is a very mild technical condition used in the proofs of some propositions. For example, the condition is satisfied when $\Phi_l^{\text{out}}(\rho_l)$ attains its maximum.

PP/FIFO Rule. *For $v \in \mathcal{V}^{sink}$,*

$$f_l^{out}(\rho) \triangleq \Phi_l^{out}(\rho_l) \quad \forall l \in \mathcal{L}_v^{in}. \tag{7}$$

For each $v \in \mathcal{V} \backslash \mathcal{V}^{sink}$, we must ensure that the inflow of each outgoing link does not exceed the link supply. To this end, define

$$\alpha^v(\rho) \triangleq \max_{\alpha\in[0,1]} \quad \alpha \tag{8}$$

$$\text{s.t.} \quad \alpha \sum_{j\in\mathcal{L}_v^{in}} \beta_{jk}^v \Phi_j^{out}(\rho_j) \le \Phi_k^{in}(\rho_k) \quad \forall k \in \mathcal{L}_v^{out}. \tag{9}$$

By scaling the demand of each link by $\alpha^v(\rho)$, we ensure that the supply of each downstream link is not violated:

$$f_l^{out}(\rho) \triangleq \alpha^v(\rho)\Phi_l^{out}(\rho_l) \quad \forall l \in \mathcal{L}_v^{in}. \tag{10}$$

To complete the model, we determine $f_l^{in}(\rho)$ from the following conservation of mass restriction on the flow functions:

$$f_l^{in}(\rho) = \sum_{k\in\mathcal{L}_{\tau(l)}^{in}} \beta_{kl}^{\tau(l)} f_k^{out}(\rho) \quad \forall l \in \mathcal{O}. \tag{11}$$

■

The format of (8) emphasizes the fact that the outflow of a link is the largest possible flow such that neither link demand nor downstream supply is exceeded and such that the outflow of all incoming links at a junction is proportional to the demand of these links.

Remark 2. *Equations (7)–(11) imply that $F_l(\cdot)$ is a function of ρ_k only if*

$$k \in \mathcal{L}_{\tau(l)}^{in} \cup \mathcal{L}_{\tau(l)}^{out} \cup \mathcal{L}_{\sigma(l)}^{in} \cup \mathcal{L}_{\sigma(l)}^{out} \tag{12}$$

and thus the network dynamics possess a local dependence property, as is expected for a reasonable traffic flow model. Furthermore, $f_l^{in}(\cdot)$ and $f_l^{out}(\cdot)$ cannot both be functions of ρ_k if $k \neq l$, that is, a link k may only directly affect the inflow or outflow of another link l, but not both.

We note that the adaptation to the CTM described briefly in [10, Section II.C] differs from our model in the following important respects: the model as discussed in [10, Section II.C] assumes a path graph network topology, requires identical links (*i.e.* identical supply and demand functions), and only considers trajectories in the region in which supply does not restrict flow (that is, $\alpha^v(\rho) = 1$ for all $v \in \mathcal{V}$ in our model), which is shown to be positively invariant given their assumptions. In this work, we generalize each of these restrictions.

D. Basic Properties of the PP/FIFO rule

We first note two properties captured by the proposed network flow model.

Lemma 1. *A simple consequence of (7)–(11) is*

$$f_l^{in}(\rho) \le \Phi_l^{in}(\rho_l) \qquad \forall l \in \mathcal{O} \tag{13}$$

$$f_l^{out}(\rho) \le \Phi_l^{out}(\rho_l) \qquad \forall l \in \mathcal{L}. \tag{14}$$

We also note that the domain $\mathcal{D}$ in (3) is positively invariant. To see this, we observe $\rho_l = 0$ implies $f_l^{\text{out}}(\rho) = 0$ which implies $\dot\rho_l \ge 0$, and $\rho_l = \rho^{\text{jam}}$ implies $f_l^{\text{in}}(\rho) = 0$ which implies $\dot\rho_l \le 0$. Furthermore, we have the following Proposition, which ensures global existence and uniqueness of solutions for piecewise continuous input flows $\{d_l(t)\}_{l\in\mathcal{R}}$ [15]:

Proposition 1. *For each $l \in \mathcal{L}$, $F_l(\rho,t)$ defined in (4) is Lipschitz continuous in ρ.*

E. Piecewise Differentiability

Assume $d_l(t) \equiv d_l$ for some constant d_l for all $l \in \mathcal{R}$ so that the dynamics are autonomous. Note that the solution of (8)–(9) is

$$\alpha^v(\rho) = \begin{cases} \min\left\{1, \min_{k\in\mathcal{L}_v^{\text{out}}}\left\{\left(\sum_{j\in\mathcal{L}_v^{\text{in}}}\beta_{jk}^v\Phi_j^{\text{out}}(\rho_j)\right)^{-1}\Phi_k^{\text{in}}(\rho_k)\right\}\right\} & \text{if } \exists l \in \mathcal{L}_v^{\text{in}} \text{ s.t. } \rho_l > 0 \\ 1 & \text{otherwise,} \end{cases} \tag{15}$$

thus $\{f_l^{\text{out}}(\rho)\}_{l\in\mathcal{L}_v^{\text{in}}}$ is uniquely defined in (10). Furthermore, by considering the finite set of functions possible for $\alpha^v(\rho)$ determined by the minimizing $k \in \mathcal{L}_v^{\text{out}}$ in (15), we conclude that $f_l^{\text{out}}(\rho)$ and thus $F_l(\rho)$ is a continuous selection of differentiable functions. Indeed, at each junction, either every outgoing link has adequate supply to accomodate the demand of incoming links, or there exists at least one link that does not have adequate supply. If more than one link does not have adequate supply, the most restrictive link determines the flow through the junction. Thus, for each $v \in \mathcal{V}$, there are $|\mathcal{L}_v^{\text{out}}| + 1$ functions possible for $\alpha^v(\rho)$ in (15). We then consider $F(\rho)$ to be selected from $\prod_{v\in\mathcal{V}}(|\mathcal{L}_v^{\text{out}}| + 1)$ *modes* of the network.

Let $\mathcal{I}$ denote an index set of these possible modes, and let $F^{(i)}(\rho)$ for $i \in \mathcal{I}$ denote the particular mode defined implicitly by the corresponding minimizers of (15) for each $v \in \mathcal{V}$. The function $F(\rho)$ is then piecewise differentiable.

III. Equilibria and stability with constant input

We now characterize the equilibria possible from the above model with constant input flow $\{d_l\}_{l\in\mathcal{R}}$. We will investigate the case where $\lim_{t\to\infty} F_l(\rho(t)) = 0$ for all $l \in \mathcal{L}$ when the input flow does not exceed a certain network capacity. When input flow does exceed network capacity, we consider the case where $\lim_{t\to\infty} F_l(\rho(t)) = 0$ for all $l \in \mathcal{O}$ and $\lim_{t\to\infty} f_l^{\text{out}}(\rho(t)) = c_l \le d_l$ for some constant c_l for all $l \in \mathcal{R}$. In the latter case, the density of some onramps (specifically, those with $c_l < d_l$) will diverge to infinity, but we will see that a meaningful definition of equilibrium nonetheless exists. From a practical point of view, such a characterization is useful, *e.g.*, during "rush hour" when the input flow of a traffic network may exceed network capacity for a limited but extended period of time.

Define

$$f_{\mathcal{O}}^{\text{in}}(\rho) \triangleq \begin{bmatrix} f_1^{\text{in}}(\rho) & \dots & f_{|\mathcal{O}|}^{\text{in}}(\rho) \end{bmatrix}' \tag{16}$$

$$f_{\mathcal{O}}^{\text{out}}(\rho) \triangleq \begin{bmatrix} f_1^{\text{out}}(\rho) & \dots & f_{|\mathcal{O}|}^{\text{out}}(\rho) \end{bmatrix}' \tag{17}$$

$$f_{\mathcal{R}}^{\text{out}}(\rho) \triangleq \begin{bmatrix} f_1^{\text{out}}(\rho) & \dots & f_{|\mathcal{R}|}^{\text{out}}(\rho) \end{bmatrix}' \tag{18}$$

for some enumeration of $\mathcal{O}$ and $\mathcal{R}$. There exists matrices A and B constructed from the collection of turn ratios $\{\beta_{lk}^v\}$ such that (11) is written as

$$f_{\mathcal{O}}^{\text{in}}(\rho) = A f_{\mathcal{O}}^{\text{out}}(\rho) + B f_{\mathcal{R}}^{\text{out}}(\rho). \tag{19}$$

A. Feasible input flows

Definition 1. The constant input flow $\{d_l\}_{l\in\mathcal{R}}$ is *feasible* if there exists density $\rho^e \triangleq \{\rho_l^{\text{e}}\}_{l\in\mathcal{L}} \in \mathcal{D}$ such that

$$f_l^{\text{out}}(\rho^{\text{e}}) = d_l \qquad \forall l \in \mathcal{R} \tag{20}$$

$$f_l^{\text{out}}(\rho^{\text{e}}) = f_l^{\text{in}}(\rho^{\text{e}}) \qquad \forall l \in \mathcal{O}. \tag{21}$$

We define $f_l^{\text{e}} \triangleq f_l^{\text{out}}(\rho^{\text{e}})$ for all $l \in \mathcal{L}$, and the set $\{f_l^{\text{e}}\}_{l\in\mathcal{L}}$ is called an *equilibrium flow*. ■

If the input flow is not feasible, it is said to be *infeasible*. It is clear that for a feasible input flow $\{d_l\}_{l\in\mathcal{R}}$, we must have for all $l \in \mathcal{R}$:

$$d_l \le \overline{\Phi}_l^{\text{out}} \quad \begin{array}{l}\text{if there exists } \rho_l^* < \infty \text{ such that} \\ \Phi_l^{\text{out}}(\rho_l) = \overline{\Phi}_l^{\text{out}} \text{ for all } \rho_l \ge \rho_l^*\end{array} \tag{22}$$

$$\text{or} \quad d_l < \overline{\Phi}_l^{\text{out}} \quad \text{if } \Phi_l^{\text{out}}(\rho_l) < \overline{\Phi}_l^{\text{out}} \text{ for all } \rho_l \in [0, \infty). \tag{23}$$

Proposition 2. *An equilibrium flow $\{f_l^e\}_{l\in\mathcal{L}}$ with corresponding equilibrium densities $\{\rho_l^e\}_{l\in\mathcal{L}}$ satisfies*

$$f_l^e \le \Phi_l^{crit} \qquad \forall l \in \mathcal{O}. \tag{24}$$

Proof: Suppose there exists $l \in \mathcal{O}$ such that $f_l^{\text{out}}(\rho) = f_l^{\text{in}}(\rho) = f_l^{\text{e}} > \Phi_l^{\text{crit}}$. Since $f_l^{\text{out}}(\rho) \le \Phi_l^{\text{out}}(\rho_l)$ for all ρ, we have $\Phi_l^{\text{out}}(\rho_l^{\text{e}}) > \Phi_l^{\text{crit}}$. But by Assumption 2, $\Phi_l^{\text{out}}(\rho_l) > \Phi_l^{\text{crit}}$ implies $\Phi_l^{\text{in}}(\rho_l) \le \Phi_l^{\text{crit}}$, which contradicts (21) since $f_l^{\text{in}}(\rho) \le \Phi_l^{\text{in}}(\rho_l)$ for all ρ. ■

Proposition 3. *Assume* (22)–(23). *An input flow $\{d_l\}_{l\in\mathcal{R}}$ is feasible if and only if*

$$(I - A)^{-1} B d \le \Phi^{crit} \tag{25}$$

where $d \triangleq \begin{bmatrix} d_1 & \dots & d_{|\mathcal{R}|} \end{bmatrix}'$, $\Phi^{crit} \triangleq \begin{bmatrix} \Phi_1^{crit} & \cdots & \Phi_{|\mathcal{O}|}^{crit} \end{bmatrix}'$, and $\le$ denotes elementwise inequality. Furthermore, for feasible input flows, the equilibrium flow $\{f_l^e\}_{l\in\mathcal{O}}$ is unique.

Definition 2. An ordinary link $l \in \mathcal{O}$ is said to be in *freeflow* if $\rho_l \le \rho_l^{\text{crit}}$. Otherwise, link l is *congested*. ■

Corollary 1. *For a feasible input flow, there exists a unique equilibrium density $\{\rho_l^e\}_{l\in\mathcal{L}}$ such that each link $l \in \mathcal{O}$ is in freeflow.*

While equilibrium flows for feasible input flows are unique, equilibrium densities are, in general, not unique. However, if the input flow is *strictly feasible* then the equilibrium density is unique:

Definition 3. A feasible input flow $\{d_l\}_{l\in\mathcal{R}}$ is said to be *strictly feasible* if the corresponding (unique) equilibrium flow satisfies $f_l^{\text{e}} < \Phi_l^{\text{crit}}$ for all $l \in \mathcal{O}$. ■

Proposition 4. *If the input flow $\{d_l\}_{l\in\mathcal{R}}$ is strictly feasible, then the corresponding equilibrium density $\{\rho_l^{\text{e}}\}_{l\in\mathcal{L}}$ is unique and each link $l \in \mathcal{O}$ is in freeflow.*

B. Infeasible input flows

We now wish to extend a notion of equilibrium to the case when the input flow is infeasible. We have shown that $\mathcal{D}$ is invariant and thus the density of an ordinary link $l \in \mathcal{O}$ will not exceed the jam density ρ_l^{jam} for any input flow. Therefore only onramps may experience density accumulation due to an infeasible input flow. We therefore define a notion of equilibrium in which the densities, input flows, and output flows on the ordinary links, and the output flows on onramp links, approach a steady state while onramp densities may grow without bound.

Definition 4. For any input flow $\{d_l\}_{l\in\mathcal{R}}$, the collection $\{f_l^{\text{e}}\}_{l\in\mathcal{L}}$ is called an *equilibrium flow* of the traffic network system if there exists a set $\{\rho_l^{\text{e}}\}_{l\in\mathcal{L}}$ with

$$0 \le \rho_l^{\text{e}} \le \rho_l^{\text{jam}} \qquad \forall l \in \mathcal{O} \tag{26}$$

$$0 \le \rho_l^{\text{e}} \le \infty \qquad \forall l \in \mathcal{R} \tag{27}$$

such that

$$f_l^{\text{e}} = f_l^{\text{out}}(\rho^{\text{e}}) = f_l^{\text{in}}(\rho^{\text{e}}) \qquad \forall l \in \mathcal{O} \tag{28}$$

$$f_l^{\text{e}} = f_l^{\text{out}}(\rho^{\text{e}}) \qquad \forall l \in \mathcal{R} \tag{29}$$

and for all $l \in \mathcal{R}$, either

$$f_l^{\text{e}} = d_l, \qquad \text{or} \qquad f_l^{\text{e}} < d_l \text{ and } \rho_l^{\text{e}} = \infty \tag{30}$$

where $\{f_l^{\text{out}}(\rho^{\text{e}})\}_{l\in\mathcal{O}}$, $\{f_l^{\text{in}}(\rho^{\text{e}})\}_{l\in\mathcal{O}}$, and $\{f_l^{\text{out}}(\rho^{\text{e}})\}_{l\in\mathcal{R}}$ solves (7)–(11) and we interpret $\Phi_l^{\text{out}}(\infty) \triangleq \overline{\Phi}_l^{\text{out}}$ for all $l \in \mathcal{R}$. By a slight abuse of nomenclature, we call $\{\rho_l^{\text{e}}\}_{l\in\mathcal{L}}$ an *equilibrium density*. ■

Definition 4 naturally extends the definition for equilibrium flow given in Definition 1 to the case when the input flow is infeasible.

Proposition 5. *For constant input flows $\{d_l\}_{l\in\mathcal{R}}$, an equilibrium flow exists.*

For general network topologies, equilibrium flows may not be unique when the input flow is infeasible. However, equilibrium flows are unique when the network graph is a polytree:

Definition 5. A *polytree* is a directed acyclic graph with exactly one undirected path between any two vertices. ■

Equivalently, a polytree is a weakly connected directed acyclic graph for which the underlying undirected graph contains no cycles.

Proposition 6. *Given constant infeasible input flow $\{d_l\}_{l\in\mathcal{R}}$. If the traffic network graph $\mathcal{G}$ is a polytree, then the equilibrium flow $\{f_l^e\}_{l\in\mathcal{L}}$ is unique.*

If the undirected traffic network does contain cycles, then it is possible that equilibrium flows may not be unique when the input flow is infeasible. Such examples with nonunique equilibrium flows are not difficult to construct.

C. Convergence

We now give results on convergence properties of the network flows. In Proposition 7, we show that the unique equilibrium density corresponding to strictly feasible input flows is asymptotically stable, and we provide an under approximation of the region of convergence in Proposition 8. We then restrict consideration to the class of traffic networks consisting of only merging junctions and, in Proposition 9, provide a global convergence result for any input demand.

Proposition 7. *For a strictly feasible input flow $\{d_l\}_{l\in\mathcal{R}}$, the unique equilibrium density $\{\rho_l^e\}_{l\in\mathcal{L}}$ is locally asymptotically stable.*

Proposition 7 can be proved by linearizing at the equilibrium, but provides no guarantees on the size of the region of attraction. The following result implies that the region of attraction includes the box $\{\rho : 0 \le \rho_l \le \rho_l^e\ \forall l \in \mathcal{L}\}$:

Proposition 8. *For a feasible input flow, all trajectories $\rho(t)$ such that $0 \le \rho_l(0) \le \rho_l^e$ for all $l \in \mathcal{L}$ converge to $\{\rho_l^e\}_{l\in\mathcal{L}}$ where $\{\rho_l^e\}_{l\in\mathcal{L}}$ is the unique equilibrium density in Corollary 1 for which all links $l \in \mathcal{O}$ are in freeflow,* i.e.

$$\lim_{t\to\infty} \rho_l(t) = \rho_l^e \qquad \textit{if} \quad 0 \le \rho_l(0) \le \rho_l^e\ \forall l \in \mathcal{L}. \tag{31}$$

Proposition 8 applies to feasible input flows that are not necessarily strictly feasible. For a specific class of networks, we conclude global convergence to the equilibrium flow:

Proposition 9. *Given constant input flow $\{d_l\}_{l\in\mathcal{R}}$. If*
(1) *$|\mathcal{L}_v^{out}| \le 1$, for all $v \in \mathcal{V}$,*
(2) *For all $v \in \mathcal{V}$, there exists Γ_v s.t. $\gamma_l = \Gamma_v\ \forall l \in \mathcal{L}_v^{in}$*
then there exists a unique equilibrium flow $\{f_l^e\}_{l\in\mathcal{L}}$ and

$$\lim_{t\to\infty} f_l^{in}(\rho(t)) = f_l^e \qquad \forall l \in \mathcal{O} \tag{32}$$

$$\lim_{t\to\infty} f_l^{out}(\rho(t)) = f_l^e \qquad \forall l \in \mathcal{L} \tag{33}$$

for any initial condition $\rho(0) \in \mathcal{D}$.

The condition $|\mathcal{L}_v^{\text{out}}| \le 1$, for all $v \in \mathcal{V}$ implies that each junction is a merging junction and consists of only one outgoing link or no outgoing links. The constraint $\gamma_l = \Gamma_v$ $\forall l \in \mathcal{L}_v^{\text{in}}$ for some Γ_v implies that the fraction of flow exiting a link that is routed off the network is the same for each incoming link at a particular junction. To prove Proposition 9, we introduce the following definition from [1]:

Definition 6. A matrix $A \in \mathbb{R}^{n\times n}$ is a *compartmental matrix* if $[A]_{ij} \ge 0$ for all $i \ne j$ and $\sum_{i=1}^n [A]_{ij} \le 0$ for all j where $[A]_{ij}$ is the ij-th entry of A. ■

Equivalently, A is a compartmental matrix if and only if A is *Metzler* [16] and $\mu_1(A) \le 0$ where $\mu_1(A) \triangleq \lim_{h\to 0^+} \frac{1}{h}(||I + hA||_1 - 1)$ is the *logarithmic norm* of A and $||A||_1$ is the matrix norm induced by the vector one-norm [17]. This observation provides a connection to *contraction theory* for non-Euclidean norms [18]. In particular, Lemma 2 shows that $F(\rho)$ is nonexpansive in a region of the state-space relative to a weighted one-norm. We have the following important general result:

Lemma 2. *Given $\Omega \subseteq \mathcal{D}$ and diagonal matrix W with positive entries on the diagonal such that $WJ^{(i)}(\rho)$ is a compartmental matrix for all $i \in \mathcal{I}$ and all $\rho \in \Omega^\circ$ such that $F^{(i)}(\rho) = F(\rho)$ where $J^{(i)}(\rho)$ denotes the Jacobian of $F^{(i)}(\rho)$ and Ω° denotes the interior of Ω. Then $V(\rho) \triangleq ||WF(\rho)||_1$ is decreasing along trajectories $\rho(t)$ of the traffic network when $\rho(t) \in \Omega$. Moreover, if Ω is positively invariant, then the flows of the network converge to an equilibrium flow as defined in Definition 4.*

Lemma 2 is proved in a similar manner to the proof of [19, Theorem 2].

We now turn our attention to the class of networks considered in Proposition 9. For networks satisfying condition 1 of Proposition 9, $\mathcal{V}^{\text{sink}}$ is a singleton, suppose $\mathcal{V}^{\text{sink}} = \{v_{\text{sink}}\}$. Furthermore, for each $l \in \mathcal{L}$ there exists a unique path $\{l_1, \dots, l_{n_l}\} \subset \mathcal{L}$ with $l_1 = l$ such that $\sigma(l_{n_l}) = v_{\text{sink}}$. Supposing 1) and 2) of Proposition 9, let

$$w_l \triangleq \begin{cases} 1 - \Gamma_{\sigma(l)} & \text{if } \Gamma_{\sigma(l)} < 1 \\ 1 & \text{if } \Gamma_{\sigma(l)} = 1 \end{cases} \qquad \forall l \in \mathcal{L} \tag{34}$$

$$W_l \triangleq w_{l_1} \cdot \dots \cdot w_{l_{n_l}}. \tag{35}$$

Lemma 3. *Given a traffic network with constant input flows $\{d_l\}_{l\in\mathcal{R}}$ satisfying the 1) and 2) of Proposition 9. Define w_l and W_l as in* (34)–(35)*, and let $W \triangleq diag(W_1, \dots, W_{|\mathcal{L}|})$. Then $W\left(\frac{\partial F^{(i)}}{\partial \rho}(\rho)\right)$ is a compartmental matrix for all $i \in \mathcal{I}$ such that $F(\rho) = F^{(i)}(\rho)$ and $\rho \in \mathcal{D}^\circ$.*

Proof:
Consider a particular link l and the corresponding lth column of $\partial F^{(i)}/\partial \rho$ for some $i \in \mathcal{I}$. In the following, we omit the superscript (i) and all partial derivatives are assumed to correspond to the mode i. Appealing to Remark 2, we have

$$\sum_{k\in\mathcal{L}} W_k \frac{\partial F_k}{\partial \rho_l} = \frac{\partial}{\partial \rho_l}\Big(-\sum_{k\in\mathcal{L}_{\tau(l)}^{\text{in}}} W_k f_k^{\text{out}} + W_l f_l^{\text{in}} - \sum_{k\in\mathcal{L}_{\sigma(l)}^{\text{in}}} W_k f_k^{\text{out}} + \sum_{k\in\mathcal{L}_{\sigma(l)}^{\text{out}}} W_k f_k^{\text{in}}\Big). \tag{36}$$

It can be shown that, for networks such that $|\mathcal{L}_v^{\text{out}}| \le 1$ for all $v \in \mathcal{V}$, we have $\frac{\partial F_k}{\partial \rho_l} \ge 0$ for all $l \ne k$, *i.e.*, the system is *cooperative* [11]. Observe that $\beta_{kl}^{\tau(l)} = (1 - \Gamma_{\tau(l)})$ and $W_k = (1 - \Gamma_{\tau(l)})W_l$ for all $k \in \mathcal{L}_{\tau(l)}^{\text{in}}$ for all $l \in \mathcal{O}$. We subsequently show

$$\frac{\partial}{\partial \rho_l}\left(-\sum_{k\in\mathcal{L}_{\tau(l)}^{\text{in}}} W_k f_k^{\text{out}}(\rho) + W_l f_l^{\text{in}}(\rho)\right)(x) = 0 \tag{37}$$

$$\frac{\partial}{\partial \rho_l}\left(-\sum_{k\in\mathcal{L}_{\sigma(l)}^{\text{in}}} W_k f_k^{\text{out}}(\rho) + \sum_{k\in\mathcal{L}_{\sigma(l)}^{\text{out}}} W_k f_k^{\text{in}}(\rho)\right)(x) \le 0 \tag{38}$$

for all $x \in \mathcal{D}^\circ$. Combining (37)–(38) with (36) gives $\sum_{k\in\mathcal{L}} W_k \frac{\partial F_k}{\partial \rho_l} \le 0$ for all l, thus proving the claim. To prove (37)–(38), consider a particular $x \in \mathcal{D}^\circ$:

(Flows at $\tau(l)$) If upstream demand exceeds the supply of link l, that is, $l \in \mathcal{O}$ and $\Phi_l^{\text{in}}(x_l) < \sum_{k\in\mathcal{L}^{\text{in}}_{\tau(l)}} \beta_{kl}^{\tau(l)} \Phi_k^{\text{out}}(x_k)$, then the PP/FIFO rule stipulates

$$f_k^{\text{out}}(y) = \Phi_l^{\text{in}}(y_l) \frac{\Phi_k^{\text{out}}(y_k)}{\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(l)}} \beta_{jl}^{\tau(l)} \Phi_j^{\text{out}}(y_j)} \ \forall k \in \mathcal{L}^{\text{in}}_{\tau(l)} \quad (39)$$

$$f_l^{\text{in}}(y) = \Phi_l^{\text{in}}(y_l) \quad (40)$$

for all $y \in \mathcal{B}_\epsilon(x)$ for some $\epsilon > 0$ where $\mathcal{B}_\epsilon(x)$ is the ball of radius ϵ centered at x. Then $\sum_{k\in\mathcal{L}^{\text{in}}_{\tau(l)}} f_k^{\text{out}}(y) = \left(1-\Gamma_{\tau(l)}\right)^{-1} \Phi_l^{\text{in}}(y_l)$ and

$$-\sum_{k\in\mathcal{L}^{\text{in}}_{\tau(l)}} W_k f_k^{\text{out}}(y) + W_l f_l^{\text{in}}(y) = 0 \ \forall y \in \mathcal{B}_\epsilon(x) \quad (41)$$

which implies (37).

If link l has adequate supply, we have $f_k^{\text{out}}(x) = \Phi_k^{\text{out}}(x_k)$ for $k \in \mathcal{L}^{\text{in}}_{\tau(l)}$ and $f_l^{\text{in}}(x) = \sum_{k\in\mathcal{L}^{\text{in}}_{\tau(l)}} \beta_{kl}^{\tau(l)} \Phi^{\text{out}}(x_k)$, neither of which is a function of x_l, and thus also (37) holds.

(Flows at $\sigma(l)$) By hypothesis, $\mathcal{L}^{\text{out}}_{\sigma(l)}$ is either empty or a singleton. If it is empty, then $f_k^{\text{out}}(x) = \Phi_k^{\text{out}}(x_k)$ for all $k \in \mathcal{L}^{\text{in}}_{\sigma(l)}$ and the lefthand side of (38) is

$$-\sum_{k\in\mathcal{L}^{\text{in}}_{\sigma(l)}} W_k \tfrac{\partial}{\partial \rho_l} f_k^{\text{out}}(x) = -W_l \tfrac{d}{d\rho_l} \Phi_l^{\text{out}}(x_l) < 0 \quad (42)$$

and (38) holds. If $\mathcal{L}^{\text{out}}_{\sigma(l)}$ is nonempty, let $\mathcal{L}^{\text{out}}_{\sigma(l)} = \{m\}$ and observe that $W_k = (1-\Gamma_{\sigma(l)}) W_m$ and $\beta_{km}^{\sigma(l)} = 1 - \Gamma_{\sigma(l)}$ for all $k \in \mathcal{L}^{\text{in}}_{\sigma(l)}$. Suppose link m has adequate supply for upstream demand so that $f_k^{\text{out}}(x) = \Phi_k^{\text{out}}(x_k)$ for all $k \in \mathcal{L}^{\text{in}}_{\sigma(l)}$ and $f_m^{\text{in}}(x) = \sum_{k\in\mathcal{L}^{\text{in}}_{\sigma(l)}} \beta_{km}^{\sigma(l)} \Phi_k^{\text{out}}(x_k)$. Then the lefthand side of (38) is $-W_l \frac{d}{d\rho_l} \Phi_l^{\text{out}}(x_l) + W_m \beta_{lm}^{\sigma(l)} \frac{d}{d\rho_l} \Phi_l^{\text{out}}(x_l) = 0$ and therefore (38) holds. If link m has inadequate supply, there exists $\epsilon > 0$ such that

$$\sum_{k\in\mathcal{L}^{\text{in}}_{\sigma(l)}} f_k^{\text{out}}(y) = \left(1-\Gamma_{\sigma(l)}\right)^{-1} \Phi_m^{\text{in}}(y_m) \quad (43)$$

$$\forall y \in \mathcal{B}_\epsilon(x) \quad (44)$$

$$f_m^{\text{in}}(y) = \Phi_m^{\text{in}}(y_m) \qquad \forall y \in \mathcal{B}_\epsilon(x). \quad (45)$$

Then $-\sum_{k\in\mathcal{L}^{\text{in}}_{\sigma(l)}} W_k f_k^{\text{out}}(y) + W_m f_m^{\text{in}}(y) = 0$ for all $y \in \mathcal{B}_\epsilon(x)$ and (38) follows. ■

Proof: [Proof of Proposition 9] A network satisfying the 1) and 2) of the proposition consists of only merging junctions and is necessarily a polytree, thus Proposition 6 ensures uniqueness of the equilibrium flow.

By Lemma 3 above, $WJ^{(i)}(\rho)$ is a compartmental matrix for all $i \in \mathcal{I}$ such that $F(\rho) = F^{(i)}(\rho)$ and all $\rho \in \mathcal{D}^\circ$. Applying Lemma 2 with $\Omega \triangleq \mathcal{D}$ completes the proof. ■

IV. Conclusions

We have proposed and analyzed a macroscopic traffic flow model that merges ideas from compartmental system theory and dynamical system theory with existing, validated traffic network models. The main purpose of this paper is to propose and characterize a general model that is amenable to future research such as studying ramp metering control and transient behavior. Early results suggest that our model is well-suited for analysis of optimal ramp metering, and we will pursue this in more depth.

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