On the Mixed Monotonicity of FIFO Traffic Flow Models

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Abstract

At diverging junctions in vehicular traffic networks, congestion on one outgoing link may impede traffic flow to other outgoing links. This phenomenon is referred to as the *first-in-first-out (FIFO)* property. Traffic network models that do not include the FIFO property result in monotone dynamics for which powerful analysis techniques exist. This note shows that a large class of FIFO network models are nonetheless *mixed monotone*. Mixed monotone systems significantly generalize the class of monotone systems and enable similarly powerful analysis techniques. The studied class of models includes the case when the FIFO property is only partial, that is, traffic flow through diverging junctions exhibit both FIFO and non-FIFO phenomena.

1 Introduction

Models of vehicular traffic flow at diverging junctions must account for the effects, if any, of congestion of one outgoing link on the flow to other outgoing links. If congestion on one outgoing link negatively impacts the incoming flow to any other outgoing link, the diverging junction is said to be a *first-in-first-out (FIFO)* node model. Otherwise, the diverging junction is said to be a *non-FIFO* node model. To the extent that FIFO node models have been studied in the literature, it is often assumed that complete congestion on one outgoing link completely blocks access to another outgoing link; we say such a diverging junction model is a *fully FIFO* node model. Of particular interest in this note is the case when the node model is FIFO but not fully FIFO, and we refer to such models as *partially FIFO*, see Figure 1. The FIFO effect in traffic flow networks has been observed even for multilane diverging junctions [1, 2].

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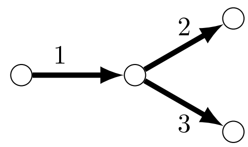

Figure 1: A traffic network consisting of a diverging junction with one incoming link and two outgoing links. When traffic flow is assumed to obey the *first-in-first-out* (FIFO) property, congestion on link 2 (resp. 3) impedes flow to link 3 (resp. 2) whereas in a non-FIFO flow model, the flow from link 1 to link 2 (resp. link 3) is independent of the congestion on link 3 (resp. 2). For the FIFO case, if complete congestion on one outgoing link completely impedes the outgoing flow from link 1, the node model at the diverging junction is said to be *fully FIFO*, otherwise it is *partially FIFO*.

Whether a node model of a diverging junction is FIFO or non-FIFO affects the qualitative dynamical behavior of traffic flow through the junction. In [3], a class of non-FIFO node models are studied. Such non-FIFO node models may be interpreted in at least two ways: 1) As a relaxation of the FIFO assumption, or 2) As a model in which vehicles always myopically reroute to avoid congested links. An attractive feature of non-FIFO node models is that the resulting traffic network dynamics are monotone, as is shown in [3]. Trajectories of a *monotone* dynamical system preserve a partial order over the system's state [4, 5]. Preservation of this partial order imposes restrictions on the behavior exhibited by such systems which is exploited for, *e.g.*, characterization of equilibria and stability analysis in [3].

In general, FIFO node models are not monotone. In [6], it is shown that a particular fully FIFO node model is *mixed monotone*, which significantly generalizes the class of monotone systems. However, a fully FIFO model may be conservative since complete congestion on one outgoing link completely blocks flow to other outgoing links. In this note, we study a general class of *partially FIFO* node models and show that the resulting dynamics are mixed monotone.

2 Network Flow Model

A traffic flow network consists of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{L})$ with *junctions* or *nodes* $\mathcal{V}$ and *links* $\mathcal{L}$. Let $\sigma(\ell)$ and $\tau(\ell)$ denote the head and tail junction of link $\ell \in \mathcal{L}$, respectively, where we assume $\sigma(\ell) \neq \tau(\ell)$, *i.e.*, no self-loops. Traffic flows from $\tau(\ell)$ to $\sigma(\ell)$.

For each $v \in \mathcal{V}$, we denote by $\mathcal{L}_v^{\text{in}} \subset \mathcal{L}$ the set of input links to node v and by $\mathcal{L}_v^{\text{out}} \subset \mathcal{L}$ the set of output links, *i.e.*,

$$\mathcal{L}_v^{\text{in}} \triangleq \{\ell \mid \sigma(\ell) = v\} \tag{1}$$

$$\mathcal{L}_v^{\text{out}} \triangleq \{\ell \mid \tau(\ell) = v\}. \tag{2}$$

For each ℓ, we denote by $\mathcal{L}_\ell^{\text{up}} \subset \mathcal{L}$ the set of links immediately upstream of link ℓ, and by $\mathcal{L}_\ell^{\text{down}} \subset \mathcal{L}$ the set of links immediately downstream of link ℓ. We say that links ℓ and k are *adjacent* if $\tau(\ell) = \tau(k)$ and $\ell \neq k$ and let $\mathcal{L}_\ell^{\text{adj}} \subset \mathcal{L}$ be the set of links adjacent to link ℓ. Thus

$$\mathcal{L}_\ell^{\text{up}} \triangleq \{k \in \mathcal{L} \mid \sigma(k) = \tau(\ell)\} = \mathcal{L}_{\tau(\ell)}^{\text{in}} \tag{3}$$

$$\mathcal{L}_\ell^{\text{down}} \triangleq \{k \in \mathcal{L} \mid \tau(k) = \sigma(\ell)\} = \mathcal{L}_{\sigma(\ell)}^{\text{out}} \tag{4}$$

$$\mathcal{L}_\ell^{\text{adj}} \triangleq \{k \in \mathcal{L} \mid \tau(k) = \tau(\ell), k \neq \ell\} = \mathcal{L}_{\tau(\ell)}^{\text{out}} \backslash \{\ell\}. \tag{5}$$

Each link $\ell \in \mathcal{L}$ has state $x_\ell(t) \geq 0$ evolving over time that is the density of vehicles on link ℓ. We denote the state of the network by $x(t) \triangleq \{x_\ell(t)\}_{\ell \in \mathcal{L}}$. Vehicles flow from link to link over time; the state-dependent flow of vehicles from link k to link ℓ is denoted by $f_{k \to \ell}(x)$. We assume $f_{k \to \ell}(x) \equiv 0$ if $\sigma(k) \neq \tau(\ell)$ so that flow is allowed only between links connected via a junction. Furthermore, vehicles flow to link ℓ from outside the network at rate $f_{\to \ell}(x)$ and vehicles leave the network from link ℓ at rate $f_{\ell \to}(x)$ so that

$$\dot{x}_\ell = \sum_{k \in \mathcal{L}} f_{k \to \ell}(x) - \sum_{j \in \mathcal{L}} f_{\ell \to j}(x) + f_{\to \ell}(x) - f_{\ell \to}(x) =: F_\ell(x). \tag{6}$$

In Section 3, we suggest specific forms for $f_{k \to \ell}$, $f_{\ell \to}$, and $f_{\to \ell}$ based on phenomenological properties of traffic flow. Here, we only assume x evolves on the invariant subspace $\mathcal{X} \subseteq (\mathbb{R}_{\geq 0})^{\mathcal{L}}$ where $\mathbb{R}_{\geq 0} = \{z \in \mathbb{R} \mid z \geq 0\}$ and for all $\ell, k \in \mathcal{L}$, we assume each $f_{\to \ell}(x)$, $f_{\ell \to}(x)$, and $f_{k \to \ell}(x)$ is locally Lipschitz continuous.

We further assume that each $f_{k \to \ell}(x)$ may be decomposed as

$$f_{k \to \ell}(x) = f_{k \to \ell}^{\text{F}}(x) + f_{k \to \ell}^{\text{NF}}(x) \tag{7}$$

where $f_{k \to \ell}^{\text{F}}(x)$ is the flow from link k to link ℓ that is subject to the FIFO phenomenon and $f_{k \to \ell}^{\text{NF}}(x)$ is the flow from link k to link ℓ that is not subject to the FIFO phenomenon.

The following captures the fundamental properties of traffic flow networks.

Assumption 1. *For all $\ell, k \in \mathcal{L}$, the functions $f_{k \to \ell}(x)$, $f_{\ell \to}(x)$, $f_{\to \ell}(x)$ are locally Lipschitz continuous. For all $x \in \mathcal{X}$ and whenever the given derivative* *exists,*

$$\frac{\partial f_{\to\ell}}{\partial x_m}(x) \geq 0 \qquad \forall \ell, m \in \mathcal{L} \textit{ such that } m \neq \ell \tag{8}$$

$$\frac{\partial f_{\ell\to}}{\partial x_m}(x) \leq 0 \qquad \forall \ell, m \in \mathcal{L} \textit{ such that } m \neq \ell \tag{9}$$

$$\frac{\partial f^{\mathrm{NF}}_{k\to\ell}}{\partial x_m}(x) \equiv 0 \qquad \forall \ell, k, m \in \mathcal{L} \textit{ such that } m \notin \mathcal{L}^{\mathrm{in}}_{\sigma(k)} \cup \mathcal{L}^{\mathrm{out}}_{\sigma(k)} \tag{10}$$

$$\frac{\partial f^{\mathrm{F}}_{k\to\ell}}{\partial x_m}(x) \equiv 0 \qquad \forall \ell, k, m \in \mathcal{L} \textit{ such that } m \notin \mathcal{L}^{\mathrm{in}}_{\sigma(k)} \cup \mathcal{L}^{\mathrm{out}}_{\sigma(k)} \tag{11}$$

$$\frac{\partial}{\partial x_m}\left(\sum_{j\in\mathcal{L}} f^{\mathrm{F}}_{j\to\ell}\right)(x) \geq 0 \qquad \forall \ell, m \in \mathcal{L} \textit{ such that } m \in \mathcal{L}^{\mathrm{up}}_{\ell} \tag{12}$$

$$\frac{\partial}{\partial x_m}\left(\sum_{j\in\mathcal{L}} f^{\mathrm{NF}}_{j\to\ell}\right)(x) \geq 0 \qquad \forall \ell, m \in \mathcal{L} \textit{ such that } m \in \mathcal{L}^{\mathrm{up}}_{\ell} \tag{13}$$

$$\frac{\partial}{\partial x_m}\left(\sum_{j\in\mathcal{L}} f_{\ell\to j}\right)(x) \leq 0 \qquad \forall \ell, m \in \mathcal{L} \textit{ such that } m \in \mathcal{L}^{\mathrm{in}}_{\sigma(\ell)} \cup \mathcal{L}^{\mathrm{out}}_{\sigma(\ell)},\ m \neq \ell \tag{14}$$

$$\frac{\partial f^{\mathrm{NF}}_{k\to\ell}}{\partial x_m}(x) \geq 0 \qquad \forall \ell, k, m \in \mathcal{L} \textit{ such that } m \in \mathcal{L}^{\mathrm{adj}}_{\ell} \tag{15}$$

$$\frac{\partial f^{\mathrm{F}}_{k\to\ell}}{\partial x_m}(x) \leq 0 \qquad \forall \ell, k, m \in \mathcal{L} \textit{ such that } m \in \mathcal{L}^{\mathrm{adj}}_{\ell}. \tag{16}$$

We have the following intuitive interpretations of (8)–(12):

- (Eq. (8)) For any $m \neq \ell$, increasing the density on link m can only increase the exogenous flow into link ℓ. For example, congestion on link m of the network may cause vehicles that wish to enter the network to reroute and enter at link ℓ.

- (Eq. (9)) For any $m \neq \ell$, increasing the density on link m can only decrease the flow that exits the network from link ℓ. For example, downstream congestion on link m may impede the outflow of vehicles via an offramp on link ℓ.

- (Eqs. (10) and (11)) The flow rate from link k to link ℓ through some junction $v = \sigma(k) = \tau(\ell) \in \mathcal{V}$ is instantaneously affected by the change in density of vehicles on link m only if m is incoming or outgoing of junction v.

- (Eqs. (12) and (13)) For any link m immediately upstream of link ℓ (that is, $\sigma(m) = \tau(\ell)$), increasing the density of vehicles on link m cannot decrease the net incoming FIFO or non-FIFO flow to link ℓ.

- (Eq. (14)) For any link $m \neq \ell$ incoming or outgoing from junction $\sigma(\ell)$, increasing the density of vehicles on link m cannot increase the net outgoing flow from link ℓ to other outgoing links.

- (Eq. (15)) For any link m adjacent to link ℓ, increasing the density of link m can only increase the non-FIFO flow from an upstream link k to ℓ. This may occur if, *e.g.*, vehicles reroute to avoid the increased congestion on link m.

- (Eq. (16)) For any link m adjacent to link ℓ, increasing the density of link m can only decrease the FIFO flow from an upstream link k to link ℓ. This captures the fundamental feature of FIFO flow whereby congestion on link m may block access to link ℓ.

Requirements (8)–(14) are standard for traffic flow networks. The requirement (15) is found in, *e.g.*, [7, Definition 2] where it is used to establish monotonicity for non-FIFO policies. Requirement (16) captures the FIFO phenomenon.

3 Examples

We now present several related examples satisfying (8)–(16) based on the *supply* and *demand* concept of traffic flow. We assume each link $\ell \in \mathcal{L}$ possesses a *jam density* $\overline{x}_\ell$ such that $x_\ell(t) \in [0, \overline{x}_\ell]$ for all time and thus $\mathcal{X} = \prod_{\ell \in \mathcal{L}} [0, \overline{x}_\ell]$. We further assume each link possesses a state-dependent *demand* function $d_\ell(x_\ell)$ and a state-dependent *supply* function $s_\ell(x_\ell)$ satisfying:

Assumption 2. *For each $\ell \in \mathcal{L}$:*

- *The demand function $d_\ell(x_\ell) : [0, \overline{x}_\ell] \to \mathbb{R}_{\geq 0}$ is strictly increasing and Lipschitz continuous with $d_\ell(0) = 0$.*

- *The supply function $s_\ell(x_\ell) : [0, \overline{x}_\ell] \to \mathbb{R}_{\geq 0}$ is strictly decreasing and Lipschitz continuous with $s_\ell(\overline{x}_\ell) = 0$.*

The demand of a link is interpreted as the maximum outflow of the link, and the supply of a link is interpreted as the maximum inflow of the link.

Let $\mathcal{R} = \{\ell \in \mathcal{L} \mid \mathcal{L}^{\text{in}}_{\tau(\ell)} = \emptyset\}$, that is, $\mathcal{R}$ is the set of links for which there are no upstream links. We assume exogenous traffic enters the network only through $\mathcal{R}$ so that

$$f_{\to\ell}(x) \equiv 0 \quad \text{for all } \ell \notin \mathcal{R}. \tag{17}$$

For each $\ell \in \mathcal{R}$, we assume there exists a constant exogenous inflow demand δ_ℓ such that

$$f_{\to\ell}(x) = \min\{\delta_\ell, s_\ell(x_\ell)\} \text{ for all } \ell \in \mathcal{R}. \tag{18}$$

We further assume that $f_{\ell\to}(x)$ is a fixed fraction γ_ℓ of the total outflow from link ℓ if there are any links downstream of ℓ, otherwise $f_{\ell\to}(x)$ is equal to the demand of link ℓ. That is,

$$f_{\ell\to}(x) = \begin{cases} \gamma_\ell \sum_{j\in\mathcal{L}} f_{\ell\to j}(x) & \text{if } \mathcal{L}^{\text{out}}_{\sigma(\ell)} \neq \emptyset \\ d_\ell(x_\ell) & \text{otherwise} \end{cases} \qquad \forall \ell \in \mathcal{L} \tag{19}$$

where $\gamma_\ell \geq 0$ for each ℓ such that $\mathcal{L}^{\text{out}}_{\sigma(\ell)} \neq \emptyset$.

Finally, we assume there exist fixed turn ratios $\beta_\ell > 0$ for each ℓ with $\mathcal{L}^{\text{up}}_\ell \neq \emptyset$ that describe how vehicles route through the network. The role of these turn ratios is made explicit subsequently, but the interpretation is that β_ℓ is the fraction of the upstream demand that is bound for link ℓ. Note that each turn ratio is associated with an outgoing link and not an incoming-outgoing link pair, thus at the upstream junction $\tau(\ell)$, we do not allow different turn ratios for the different incoming links. Physically, this means that traffic from all incoming links join at the junction and a fraction β_ℓ of the demand is destined for link ℓ.

It remains to characterize $f_{k\to\ell}(x)$ for all $\ell, k \in \mathcal{L}$.

Example 1 (non-FIFO)**.** *For all $\ell \in \mathcal{L}$, let*

$$\alpha^{\text{NF}}_\ell(x) = \min\left\{1, \frac{s_\ell(x_\ell)}{\beta_\ell \sum_{j\in\mathcal{L}^{\text{up}}_\ell} d_j(x_j)}\right\}. \tag{20}$$

Let $f^{\text{F}}_{k\to\ell}(x) \equiv 0$ for all $k, \ell \in \mathcal{L}$ and let

$$f^{\text{NF}}_{k\to\ell}(x) = \alpha^{\text{NF}}_\ell(x)\beta_\ell d_k(x_k) \quad \forall \ell \in \mathcal{L},\ \forall k \in \mathcal{L}^{\text{up}}_\ell. \tag{21}$$

Example 2 (Fully FIFO)**.** *For all $v \in \mathcal{L}$, let*

$$\alpha^{\text{F}}_v(x) = \min\left\{1, \min_{k\in\mathcal{L}^{\text{out}}_v}\left\{\frac{s_k(x_k)}{\beta_k \sum_{j\in\mathcal{L}^{\text{in}}_v} d_j(x_j)}\right\}\right\}. \tag{22}$$

Let $f^{\text{NF}}_{k\to\ell}(x) \equiv 0$ for all $k, \ell \in \mathcal{L}$ and let

$$f^{\text{F}}_{k\to\ell}(x) = \alpha^{\text{F}}_{\tau(\ell)}(x)\beta_\ell d_k(x_k) \quad \forall \ell \in \mathcal{L},\ \forall k \in \mathcal{L}^{\text{up}}_\ell. \tag{23}$$

Example 3 (Convex combination of non-FIFO and fully FIFO)**.** *Let $\alpha^{\text{NF}}_\ell(x)$ be given as in* (20) *and let $\alpha^{\text{F}}_v(x)$ be given as in* (22)*. Suppose there exists $\eta_\ell \in [0,1]$ for all $\ell \in \mathcal{L}$, and let*

$$f^{\text{F}}_{k\to\ell}(x) = \eta_\ell \alpha^{\text{F}}_{\tau(\ell)}(x)\beta_\ell d_k(x_k) \quad \forall \ell \in \mathcal{L},\ \forall k \in \mathcal{L}^{\text{up}}_\ell \tag{24}$$

$$f^{\text{NF}}_{k\to\ell}(x) = (1-\eta_\ell)\alpha^{\text{NF}}_\ell(x)\beta_\ell d_k(x_k) \quad \forall \ell \in \mathcal{L},\ \forall k \in \mathcal{L}^{\text{up}}_\ell. \tag{25}$$

Example 3 is proposed in [3, Example 4] and is a natural extension of the ideas in Examples 1 and 2, however it exhibits the following property: it is possible for $\alpha^{\text{NF}}_\ell(x) < 1$ yet $\sum_{j\in\mathcal{L}} f_{j\to\ell}(x) < s_\ell(x_\ell)$, that is, the supply of link ℓ

restricts the flow to link ℓ, yet the total inflow of link ℓ is less than this supply. This property may be undesirable, depending on the specific phenomena which the node model is desired to capture. We now suggest an alternative partially FIFO model inspired by the physical division of an incoming link at a diverging junction. To fix ideas, we assume that each diverging junction has exactly one incoming link, that is,

$$|\mathcal{L}_v^{\text{out}}| > 1 \implies |\mathcal{L}_v^{\text{in}}| = 1 \quad \forall v \in \mathcal{V}. \tag{26}$$

Example 4 (Shared and exclusive lanes for a partially FIFO model)**.** *Assume (26) holds. We consider $\eta_\ell \in [0,1]$ for each $\ell \in \mathcal{L}$ representing the degree of influence on link ℓ of the FIFO restriction at the intersection so that η_ℓ is the fraction of traffic bound for link ℓ that is subject to a FIFO restriction and $(1-\eta_\ell)$ is the fraction of traffic bound for link ℓ that is not subject to a FIFO restriction. For example, $1-\eta_\ell$ is the fraction of lanes at the diverging junction exclusively bound for link ℓ and η_ℓ is the fraction of lanes that are shared among all outgoing links.*

Whenever $\mathcal{L}_\ell^{\text{adj}} = \emptyset$, we assume $\eta_\ell = 1$ without loss of generality. Let $\alpha_v^{\text{F}}(x)$ be given as in (22) for all $v \in \mathcal{V}$. For $\ell \in \mathcal{L}$ such that $\mathcal{L}_\ell^{\text{adj}} \neq \emptyset$, let k be the unique link such that $\mathcal{L}_\ell^{\text{up}} = \{k\}$ (uniqueness is guaranteed by (26)), and let

$$f_{k\to\ell}^{\text{F}}(x) = \eta_\ell \alpha_{\tau(\ell)}^{\text{F}}(x)\beta_\ell d_k(x_k) \tag{27}$$

$$f_{k\to\ell}^{\text{NF}}(x) = \min\left\{(1-\eta_\ell)\beta_\ell d_k(x_k), s_\ell(x_\ell) - f_{k\to\ell}^{\text{F}}(x)\right\}. \tag{28}$$

For $\ell \in \mathcal{L}$ such that $\mathcal{L}_\ell^{\text{adj}} = \emptyset$, we have that $\eta_\ell = 1$ so that

$$f_{k\to\ell}^{\text{F}}(x) = \alpha_{\tau(\ell)}^{\text{F}}(x)\beta_\ell d_k(x_k) \quad \forall k \in \mathcal{L}_\ell^{\text{up}} \tag{29}$$

$$f_{k\to\ell}^{\text{NF}}(x) \equiv 0. \tag{30}$$

We extend Example 4 to the case where there are multiple sets of interacting outgoing links that result in a collection of FIFO restrictions.

Example 5. *Assume (26) holds. For each $v \in \mathcal{V}$ with $\mathcal{L}_v^{\text{in}} \neq \emptyset$, let $\Phi(v) \subset P(\mathcal{L}_v^{\text{in}})$, where $P(\cdot)$ denotes the power set operator, be a collection of subsets of $\mathcal{L}_v^{\text{in}}$ so that each $\varphi \in \Phi(v)$, $\varphi \subseteq \mathcal{L}_v^{\text{in}}$ is a set links which are mutually governed by a FIFO restriction. When $|\mathcal{L}_v^{\text{out}}| = 1$, we assume $\Phi(v) = \{\mathcal{L}_v^{\text{out}}\}$.*

For $\varphi \in \Phi(v)$ and $\ell \in \mathcal{L}_v^{\text{out}}$, let $\eta_{\ell,\varphi} \in [0,1]$ represent the degree of influence on link ℓ of the FIFO restriction set φ. We make the following assumptions:

$$\ell \notin \varphi \implies \eta_{\ell,\varphi} = 0 \quad \forall \varphi \in \Phi(\tau(\ell)) \tag{31}$$

$$\sum_{\varphi \in \Phi(v)} \eta_{\ell,\varphi} \leq 1 \quad \forall \ell \in \mathcal{L}_v^{\text{in}}\ \forall v \in \mathcal{V}. \tag{32}$$

Define

$$\bar{\eta}_\ell = 1 - \sum_{\varphi \in \Phi(v)} \eta_{\ell,\varphi}. \tag{33}$$

For all $v \in \mathcal{V}$ such that $|\mathcal{L}_v^{\text{out}}| > 1$, define

$$\alpha_\varphi(x) = \min\left\{1, \min_{j\in\varphi}\left\{\frac{s_j(x_j)}{\beta_j d_k(x_k)}\right\}\right\} \qquad \forall \varphi \in \Phi(v) \tag{34}$$

where k is the unique upstream link such that $\mathcal{L}_v^{\text{in}} = \{k\}$.

For $\ell \in \mathcal{L}$ such that $\mathcal{L}_\ell^{\text{adj}} \neq \emptyset$, let k be the unique link such that $\mathcal{L}_\ell^{\text{up}} = \{k\}$, and let

$$f_{k\to\ell}^{\varphi}(x) = \eta_{\ell,\varphi}\alpha_\varphi(x)\beta_\ell d_k(x_k) \quad \forall \varphi \in \Phi(\tau(\ell)) \tag{35}$$

$$f_{k\to\ell}^{\text{F}}(x) = \sum_{\varphi\in\Phi(\tau(\ell))} f_{k\to\ell}^{\varphi}(x) \tag{36}$$

$$f_{k\to\ell}^{\text{NF}}(x) = \min\left\{\bar{\eta}_\ell\beta_\ell d_k(x_k), s_\ell(x_\ell) - f_{k\to\ell}^{\text{F}}(x)\right\}. \tag{37}$$

For $\ell \in \mathcal{L}$ such that $\mathcal{L}_\ell^{\text{adj}} = \emptyset$, we again let

$$f_{k\to\ell}^{\text{F}}(x) = \alpha_{\tau(\ell)}^{\text{F}}(x)\beta_\ell d_k(x_k) \quad \forall k \in \mathcal{L}_\ell^{\text{up}} \tag{38}$$

$$f_{k\to\ell}^{\text{NF}}(x) \equiv 0 \tag{39}$$

where $\alpha_{\tau(\ell)}^{\text{F}}(x)$ is as given in (22).

Taking $\Phi(v) = \{\mathcal{L}_v^{\text{out}}\}$ for all $v \in \mathcal{V}$, we see that Example 4 is a special case of Example 5.

All the examples above satisfy

$$f_{k\to\ell}(x) \leq \beta_\ell d_k(x_k) \quad \forall k \in \mathcal{L}\ \forall \ell \in \mathcal{L}_\ell^{\text{down}} \tag{40}$$

$$f_{k\to\ell}(x) \leq s_\ell(x_\ell) \qquad \forall k \in \mathcal{L}\ \forall \ell \in \mathcal{L}_\ell^{\text{down}}. \tag{41}$$

If we further assume that $\sum_{k\in\mathcal{L}_\ell^{\text{down}}} \beta_k \leq 1$ and

$$\gamma_\ell \leq \frac{1}{\sum_{k\in\mathcal{L}_\ell^{\text{down}}} \beta_k} - 1 \tag{42}$$

for all ℓ such that $\mathcal{L}_{\sigma(\ell)}^{\text{out}} \neq \emptyset$, we have that

$$\sum_{k\in\mathcal{L}} f_{\ell\to k}(x) + f_{\ell\to}(x) \leq d_\ell(x_\ell) \quad \forall \ell \in \mathcal{L}\ \forall x \in \mathcal{X}. \tag{43}$$

Proposition 1. *Examples 1, 2, 3, 4, and 5 satisfy Assumption 1.*

Proof. It follows straightforwardly from results in [3] that the conditions of Assumption 1 hold for Example 1, and, similarly, it follows from results in [6] that the assumption holds for Example 2. From Example 1 and Example 2, the assumption immediately holds for Example 3. We now show that Example 5 satisfies Assumption 1, from which it follows that also Example 4 satisfies Assumption 1 because Example 4 is a special case of Example 5. To that end, we now prove each of condition (8)–(16) for Example 5.

- (Condition 8). Follows trivially from (17) and (18).
- (Condition (9)). Follows from (19) and Condition (14), proved below, as well as Conditions (10) and (11), proved below.
- (Conditions (10) and (11)). Follows immediately from the fact that for all $v \in \mathcal{V}$ and all $\varphi \in \Phi(v)$, $\alpha_v^{\mathrm{F}}(x)$ and $\alpha_\varphi(x)$ are only functions of $d_\ell(x_\ell)$ for $\ell \in \mathcal{L}_v^{\mathrm{in}}$ and $s_\ell(x_\ell)$ for $\ell \in \mathcal{L}_v^{\mathrm{out}}$ and from (37).
- (Conditions (12) and (13)). Consider $\ell \in \mathcal{L}$. If $|\mathcal{L}_\ell^{\mathrm{up}}| > 1$, then $\mathcal{L}_\ell^{\mathrm{adj}} = \emptyset$ by (26) and

$$\sum_{j \in \mathcal{L}} f_{j \to \ell}^{\mathrm{F}}(x) \in \left\{ \sum_{j \in \mathcal{L}_\ell^{\mathrm{up}}} \beta_\ell d_j(x_j), s_\ell(x_\ell) \right\} \tag{44}$$

$$\sum_{j \in \mathcal{L}} f_{j \to \ell}^{\mathrm{NF}}(x) \equiv 0 \tag{45}$$

and thus (12) and (13) hold.

Now suppose $|\mathcal{L}_\ell^{\mathrm{up}}| = 1$ and let $\mathcal{L}_\ell^{\mathrm{up}} = \{k\}$ so that

$$\sum_{j \in \mathcal{L}} f_{j \to \ell}^{\mathrm{F}}(x) = f_{k \to \ell}^{\mathrm{F}}(x) = \sum_{\varphi \in \Phi(\tau(\ell))} f_{k \to \ell}^{\varphi}(x) \tag{46}$$

$$\sum_{j \in \mathcal{L}} f_{j \to \ell}^{\mathrm{NF}}(x) = f_{k \to \ell}^{\mathrm{NF}}(x). \tag{47}$$

We have

$$\alpha_\varphi(x) d_k(x_k) = \min \left\{ d_k(x_k), \min_{j \in \varphi} \left\{ \frac{s_j(x_j)}{\beta_j} \right\} \right\} \tag{48}$$

and thus (12) holds by (35) and (46).

Still supposing $|\mathcal{L}_\ell^{\mathrm{up}}| = 1$ with $\mathcal{L}_\ell^{\mathrm{up}} = \{k\}$, consider now Condition (13). The only possibility for which this condition would not hold is if $\frac{\partial}{\partial x_k} f_{k \to \ell}^{\mathrm{F}}(x) > 0$ and $s_\ell(x_\ell) - f_{k \to \ell}^{\mathrm{F}}(x)$ is the minimizer in (37). But

$$\frac{\partial}{\partial x_k} f_{k \to \ell}^{\mathrm{F}}(x) > 0 \tag{49}$$

only if $\alpha_\varphi(x) = 1$ for some φ for which $\ell \in \varphi$ on some neighborhood of x so that, in particular, $s_\ell(x_\ell)/(\beta_\ell d_k(x_k)) > 1$, equivalently, $s_\ell(x_\ell) > \beta_\ell d_k(x_k)$. In this case, since $f_{k \to \ell}^{\mathrm{F}}(x) \leq \sum_{\varphi \in \Phi(\tau(\ell))} \eta_{\ell,\varphi} \beta_\ell d_k(x_k)$, we have that

$$s_\ell(x_\ell) - f_{k \to \ell}^{\mathrm{F}}(x) \geq \bar{\eta}_\ell \beta_\ell d_k(x_k), \tag{50}$$

i.e., $\bar{\eta}_\ell \beta_\ell d_k(x_k)$ is the minimizer in (37) and thus (13) holds.

- (Condition (14)). Suppose $v \in \mathcal{V}$ is such that $|\mathcal{L}_v^{\text{out}}| = 1$ and let $\mathcal{L}_v^{\text{out}} = \{k\}$. Consider $\ell \in \mathcal{L}_v^{\text{in}}$. Then

$$\sum_{j\in\mathcal{L}} f_{\ell\to j}(x) = f_{\ell\to k}^{\text{F}} = \alpha_v^{\text{F}}(x)\beta_k d_\ell(x_\ell) \tag{51}$$

$$= \min\left\{\beta_k d_\ell(x_\ell), \frac{d_\ell(x_\ell)}{\sum_{j\in\mathcal{L}_v^{\text{in}}} d_j(x_j)} s_k(x_k)\right\}. \tag{52}$$

Since $\frac{\partial}{\partial x_m}\frac{d_\ell(x_\ell)}{\sum_{j\in\mathcal{L}_v^{\text{in}}} d_j(x_j)} \leq 0$ for all $m \in \mathcal{L}_v^{\text{in}}$ with $m \neq \ell$, and $\frac{ds_k}{dx_k}(x_k) \leq 0$ (14) holds.

Now suppose $|\mathcal{L}_v^{\text{out}}| > 1$ so that $|\mathcal{L}_v^{\text{in}}| = 1$, and let $\mathcal{L}_v^{\text{in}} = \{\ell\}$. From (35)–(37), for all $j \in \mathcal{L}_v^{\text{out}}$ we have

$$f_{\ell\to j}(x) = f_{\ell\to j}^{\text{F}}(x) + f_{\ell\to j}^{\text{NF}}(x) \in \left\{f_{\ell\to j}^{\text{F}}(x) + \bar{\eta}_\ell\beta_\ell d_k(x_k), s_j(x_j)\right\}. \tag{53}$$

Consider some $\varphi \in \Phi(v)$. Then

$$\alpha_\varphi(x)d_\ell(x_\ell) = \min\left\{d_\ell(x_\ell), \min_{i\in\varphi}\left\{\frac{s_i(x_i)}{\beta_i}\right\}\right\} \tag{54}$$

so that $\frac{\partial}{\partial x_m}(\alpha_\varphi(x)d_\ell(x_\ell)) \leq 0$ for all $m \in \mathcal{L}_v^{\text{out}}$. Therefore,

$$\frac{\partial f_{\ell\to j}^{\text{F}}}{\partial x_m}(x) = \frac{\partial}{\partial x_m}\left(\sum_{\varphi\in\Phi(v)} \eta_{j,\varphi}\alpha_\varphi(x)\beta_j d_\ell(x_\ell)\right) \leq 0 \tag{55}$$

for all $m \in \mathcal{L}_v^{\text{out}}$. From (53), we have that $\frac{\partial f_{\ell\to j}}{\partial x_m}(x) \leq 0$ for all $m \in \mathcal{L}_v^{\text{out}}$ so that (14) holds. Finally, Condition (14) holds trivially when $\mathcal{L}_v^{\text{out}} = \emptyset$ for all $\ell \in \mathcal{L}_v^{\text{in}}$.

- (Condition (15)). Follows from Condition (16) below and (37).

- (Condition (16)). Follows from (35), (54), and the fact that $\frac{ds_i}{dx_i}(x_i) \leq 0$ for all $i \in \mathcal{L}$.

□

4 Mixed Monotonicity of Traffic Flow

Definition 1 (Mixed Monotone)**.** *The system $\dot{x} = G(x)$, $x \in X \subseteq \mathbb{R}^n$ where X has convex interior and G is locally Lipschitz is* mixed monotone *if there exists a locally Lipschitz continuous function $g(x, y)$ satisfying:*

1. *$g(x, x) = G(x)$ for all $x \in X$*

2. $\dfrac{\partial g_i}{\partial x_j}(x,y) \geq 0$ *for all* $x, y \in X$ *and all* $i \neq j$ *whenever the derivative exists*

3. $\dfrac{\partial g_i}{\partial y_j}(x,y) \leq 0$ *for all* $x, y \in X$ *and all* i, j *whenever the derivative exists.*

The function $g(x,y)$ *is called a* decomposition function *for the system.*

Theorem 1. *The traffic flow network model* (6) *satisfying Assumption 1 is mixed monotone.*

Proof. We construct an appropriate decomposition function $g(x,y)$. For each $\ell \in \mathcal{L}$, let

$$z^\ell(x,y) : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \tag{56}$$

be defined elementwise as

$$z_k^\ell(x,y) = \begin{cases} y_k & \text{if } k \in \mathcal{L}_\ell^{\text{adj}} \\ x_k & \text{else} \end{cases} \quad \forall \ell \in \mathcal{L}. \tag{57}$$

Define

$$\begin{aligned} g_\ell(x,y) = \sum_{k \in \mathcal{L}} \left(f^{\text{F}}_{k \to \ell}(z^\ell(x,y)) + f^{\text{NF}}_{k \to \ell}(x) \right) - \sum_{j \in \mathcal{L}} \left(f^{\text{F}}_{\ell \to j}(x) + f^{\text{NF}}_{\ell \to j}(x) \right) \\ + f_{\to \ell}(x) - f_{\ell \to}(x) \end{aligned} \tag{58}$$

and let $g(x,y) = \{g_\ell(x,y)\}_{\ell \in \mathcal{L}}$. It is immediate that $g_\ell(x,x) = F_\ell(x)$ given in (6) for all $\ell \in \mathcal{L}$.

We first show

$$\frac{\partial g_\ell}{x_m}(x,y) \geq 0 \text{ for all } m \neq \ell. \tag{59}$$

To this end, we show

$$\frac{\partial}{\partial x_m} \left(\sum_{k \in \mathcal{L}} \left(f^{\text{F}}_{k \to \ell}(z^\ell(x,y)) + f^{\text{NF}}_{k \to \ell}(x) \right) \right) \geq 0 \quad \forall m \neq \ell \tag{60}$$

$$\frac{\partial}{\partial x_m} \left(\sum_{j \in \mathcal{L}} \left(f^{\text{F}}_{\ell \to j}(x) + f^{\text{NF}}_{\ell \to j}(x) \right) \right) \leq 0 \quad \forall m \neq \ell \tag{61}$$

which, combined with (8) and (9) of Assumption 1, proves (59). We have that (61) holds for all $m \in \mathcal{L}^{\text{in}}_{\sigma(\ell)} \cup \mathcal{L}^{\text{out}}_{\sigma(\ell)}$ with $m \neq \ell$ by (14), and (10)–(11) ensures that (61) holds with equality for all $m \notin \mathcal{L}^{\text{in}}_{\sigma(\ell)} \cup \mathcal{L}^{\text{out}}_{\sigma(\ell)}$. For $m \in \mathcal{L}^{\text{up}}_\ell$, (60) holds from (12) and (13). For $m \in \mathcal{L}^{\text{adj}}_\ell$, we have $\frac{\partial}{\partial x_m}(f^{\text{F}}_{k \to \ell}(z^\ell(x,y)) = 0$ for all k by (57), and $\frac{\partial}{\partial x_m}(f^{\text{NF}}_{k \to \ell}(x)) \geq 0$ by (15), satisfying (60). For $m \notin \mathcal{L}^{\text{adj}}_\ell \cup \mathcal{L}^{\text{up}}_\ell$, we have (60) holds with equality by (10)–(11).

We now show

$$\frac{\partial g_\ell}{y_m}(x,y) \leq 0 \text{ for all } m \neq \ell. \tag{62}$$

We have that (62) holds trivially for all $m \notin \mathcal{L}_\ell^{\mathrm{adj}}$. For $m \in \mathcal{L}_\ell^{\mathrm{adj}}$, we have

$$\frac{\partial g_\ell}{\partial y_m}(x,y) = \frac{\partial}{\partial y_m}\left(\sum_{j\in\mathcal{L}} f^{\mathrm{F}}_{j\to\ell}(z^\ell(x,y))\right) \tag{63}$$

$$\leq 0 \tag{64}$$

where the inequality follows by (16). □

We remark that a sufficient condition for mixed monotonicity of $\dot{x} = G(x)$ is for each off-diagonal entry of the Jacobian matrix $\frac{\partial G}{\partial x}$ to be sign-stable over the domain $\mathcal{X}$, that is, either $\frac{\partial G_i}{\partial x_j}(x) \geq 0$ for all $x \in \mathcal{X}$ or $\frac{\partial G_i}{\partial x_j}(x) \leq 0$ for all $x \in \mathcal{X}$ for all $i \neq j$. This condition is proved for the discrete-time case in [8] and the proof for the continuous-time case is similar. In general, partially FIFO models do not satisfy this condition; this is attributable to the different sign conditions in (15) and (16) whereby an increase on some link $k \in \mathcal{L}_\ell^{\mathrm{adj}}$ may increase the non-FIFO flow to link ℓ and decrease the FIFO flow to link ℓ. Thus we require a different construction for the decomposition function as shown in the proof of Theorem 1.

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