Offset Optimization For a Network of Signalized Intersections via Semidefinite Relaxation

Samuel Coogan, Gabriel Gomes, Eric S. Kim, Murat Arcak, Pravin Varaiya

***Abstract*— We consider the problem of coordinating the traffic signals in a network of signalized intersections to reduce accumulated queues of vehicles throughout the network. We assume that all signals have a common cycle time and a fixed actuation plan, and we propose an approach for optimizing the relative phase offsets. Unlike existing techniques, our approach accommodates networks with arbitrary topology and scales well. This is accomplished by proposing a sinusoidal approximation of the queueing processes in the network, which enables a semidefinite relaxation of the offset optimization problem that is easily solved. We demonstrate the result in a case study of a traffic network in Arcadia, California.**

I. Introduction

Systematic approaches to selecting the parameters of traffic signal controllers—cycle time, green times, and intersection-to-intersection offsets—have been a subject of research in the transportation community for several decades. Here we focus on the problem of selecting offsets for a network of signalized intersections to allow platoons of vehicles to move through the network with the least hindrance by red lights. Many existing algorithms for offset optimization focus on one- or two-way arterial roads [1], [2]. Perhaps the first algorithm developed for general networks was the Generalized Combination Method of [3], in which link delay functions were combined into a network delay function using a set of network reduction rules. The mixed-integer formulation of [1] was generalized to grid networks in [4] by adding a "loop constraint" and further extended in [5] to provide computational advantages via a network decomposition technique. These methods suffer from the computational burden of mixed-integer programs and therefore are not scalable.

In this paper, we formulate the offset optimization problem as a semidefinite program. We approximate platoons of vehicles arriving at and departing from traffic signals as sinusoidal functions of time and consider minimizing the average queue lengths at intersections. The resulting optimization problem is a non-convex, quadratically constrained quadratic program (QCQP) that is well-known to be amenable to semidefinite relaxation [6]. This QCQP formulation and its relaxation is especially analogous to recent approaches to the optimal power flow problem [7], [8] and the angular synchronization problem [9], [10].

S. Coogan, E. Kim, M. Arcak, and P. Varaiya are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, {scoogan,eskim, arcak, varaiya}@eecs.berkeley.edu. G. Gomes is with the Partners for Advanced Transportation Technology, gomes@path.berkeley.edu.
This work was supported in part by California Department of Transportation (Caltrans) and in part by NSF Grant CNS-1446145.

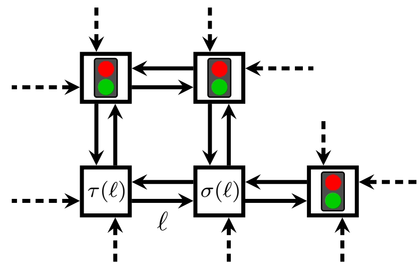

Fig. 1. A standard network of signalized intersections along with the notation used. The nodes denote the intersections. For a link $\ell \in \mathcal{L}$, $\sigma(\ell)$ denotes the signal that actuates ℓ and $\tau(\ell)$ denotes the signal immediately upstream of link ℓ. Entry links $\mathcal{E}$ are shown as dashed links. For all $\ell \in \mathcal{E}$, $\tau(\ell) = \epsilon$ where the symbol ϵ denotes the absence of an upstream signal. There is no need to model links that exit the network.

We apply our technique to a case study arterial network in Arcadia, California which is part of the Connected Corridors project at UC Berkeley [11]. The network contains thirteen signalized intersections, and simulation results show that our offset optimization technique reduces queue lengths by 27% and travel time by 8.9%. While this case study is of modest size, the semidefinite formulation proposed here easily accommodates networks with hundreds of intersections.

The paper is organized as follows. Section II introduces notational preliminaries. Section III presents the problem formulation and Section IV proposes an approach to modeling traffic flow and queue evolution using sinusoids. This approach leads to the offset optimization problem studied in Section V. We consider a case study in Section VI and provide concluding remarks in Section VII.

II. Preliminaries

We define $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$. For a periodic function $x(t)$ with period T, $\int_T x(t)dt$ denotes integration over any time interval of length T. For $a, b \in \mathbb{R}$, $\arctan(b/a) \in [0, 2\pi]$ is understood to be the four-quadrant arctangent of b/a, that is, $\arctan(b/a)$ is equal to the angle θ such that $\sin(\theta) = b/\sqrt{b^2 + a^2}$ and $\cos(\theta) = a/\sqrt{b^2 + a^2}$. For a real, square matrix $M \in \mathbb{R}^{n \times n}$, $\mathbf{Tr}(M)$ denotes the trace of M, $\mathbf{rank}(M)$ denotes the rank of M, M^T denotes the transpose of M, and M is said to be *positive semidefinite*, denoted by $M \succeq 0$, if $z^T M z \geq 0$ for all $z \in \mathbb{R}^n$. $M[i, j]$ denotes the (i, j)-th entry of M.

III. Problem Formulation

We consider a traffic network consisting of a set $\mathcal{S}$ of signalized intersections and a set $\mathcal{L}$ of *links*. A subset $\mathcal{E} \subset \mathcal{L}$ of links are *entry* links and direct exogenous traffic onto the

Parameter	In Seconds	In Radians
Cycle time	T	2π
Offset of arrival on $\ell \in \mathcal{L}$	Φ_ℓ	φ_ℓ
Phase offset of $s \in \mathcal{S}$	Θ_s	θ_s
Green split offset of $\ell \in \mathcal{L}$	Γ_ℓ	γ_ℓ
Actuation offset of $\ell \in \mathcal{L}$	$\Theta_{\sigma(\ell)} + \Gamma_\ell$	$\theta_{\sigma(\ell)} + \gamma_\ell$
Queueing offset of $\ell \in \mathcal{L}$	Ξ_ℓ	ξ_ℓ
Travel time of $\ell \in \mathcal{L}$	Λ_ℓ	λ_ℓ

TABLE I

PARAMETERS FOR THE PERIODIC, SINUSOIDAL APPROXIMATION OF ARRIVALS, DEPARTURES, AND QUEUES.

network. Let $\sigma : \mathcal{L} \to \mathcal{S}$ map each link to the signal that actuates the queue on that link, that is, $\sigma(\ell) = s \in \mathcal{S}$ if and only if signal s actuates the queue on link ℓ. Symmetrically, let $\tau : \mathcal{L} \to \mathcal{S} \cup \epsilon$ map each link to the signal immediately upstream of the link where the symbol ϵ denotes that no upstream signal exists; that is, signal $\tau(\ell)$ controls the flow of vehicles to link ℓ and we have $\ell \in \mathcal{E}$ if and only if $\tau(\ell) = \epsilon$. See Fig. 1 for an illustration of the notation.

Each signal at an intersection employs a *fixed time* control strategy whereby a repeating sequence of nonconflicting links are actuated for a fixed amount of time. The length of time, in seconds, of this sequence is the *cycle time* of the intersection.

Assumption 1 (Common Cycle Time)**.** *The signals at all intersections have common cycle time T,* i.e.*, a common frequency $\omega = 2\pi/T$.*

Table I collects notation for parameters introduced subsequently that arise from this fixed cycle time.

Each link $\ell \in \mathcal{L}$ possesses a queue with length $q_\ell(t) \in \mathbb{R}_{\geq 0}$ that evolves over time. We adopt a fluid queue model so that the queue length, arrivals, and departures are real-valued functions. Specifically, vehicles arrive at a queue from upstream links or from outside the network according to an arrival process $a_\ell(t)$ and depart the queue according to a departure process $d_\ell(t)$ which depends on the signal actuation. We describe how these processes are approximated under certain periodicity assumptions in Section IV. The queue length $q_\ell(t)$ then obeys the dynamics

$$\dot{q}_\ell(t) = a_\ell(t) - d_\ell(t). \tag{1}$$

During a cycle, we assume that each link is actuated for one contiguous interval of time (modulo a cycle length) during which vehicles depart the queue. For a signal $s \in \mathcal{S}$, the actuation durations for the upstream links $\{\ell \mid \sigma(\ell) = s\}$ and their sequencing are assumed to be fixed.

The *phase offset* Θ_s for signal $s \in \mathcal{S}$ is the offset, measured in seconds, of the actuation sequence from some global clock and is the design parameter considered here. For each link $\ell \in \mathcal{L}$, let the *green split* $\Gamma_\ell \leq T$ be the time difference of the *midpoint* of the actuation time for link ℓ and the beginning of the offset $\Theta_{\sigma(\ell)}$. It follows that

$$t = nT + (\Theta_s + \Gamma_\ell), \quad n = 0, 1, 2, \ldots \tag{2}$$

are the time instants of the midpoint of the actuation times for each link ℓ with $\sigma(\ell) = s$. We let $\theta_s = \Theta_s\omega$ and $\gamma_\ell = \Gamma_\ell\omega$ denote the normalized phase offset at signal s and the normalized green split at link ℓ, respectively. Without loss of generality, we interpret $\Theta_\epsilon = 0$.

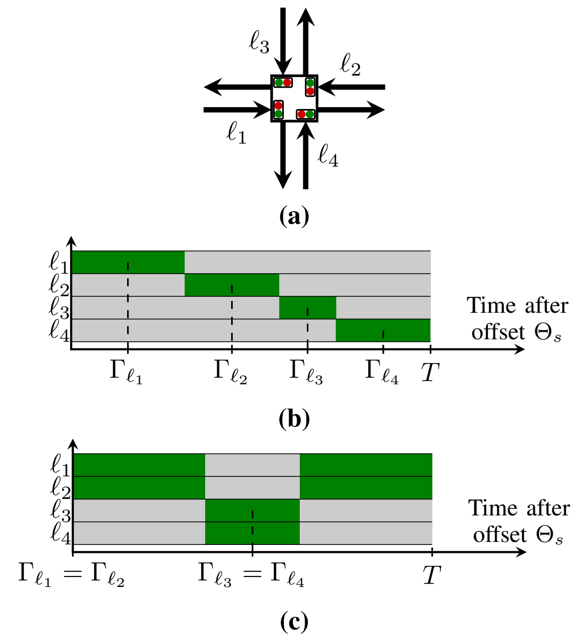

Fig. 2. **(a)** A standard four-way intersection s with incoming links $\ell_1, \ell_2, \ell_3, \ell_4$. The cycle time is divided among the incoming links according to the green splits at the intersection, which are assumed fixed. The phase offset Θ_s for the intersection is a design parameter. **(b)** Green splits when each link is actuated sequentially. The green/dark regions denote the time intervals when a link is actuated and queued vehicles are able to move through the intersection. **(c)** Green splits when links ℓ_1 and ℓ_2 are actuated simultaneously, then links ℓ_3 and ℓ_4 are actuated simultaneously. Without loss of generality, we take $\Gamma_{\ell_1} = \Gamma_{\ell_2} = 0$ and $\Gamma_{\ell_3} = \Gamma_{\ell_4} = T/2$.

Example. *Consider signal $s \in \mathcal{S}$ with four input links $\ell_1, \ell_2, \ell_3,$ and ℓ_4 as in Fig. 2(a). A possible set of green splits is shown in Fig. 2(b). The green bars indicate the time interval, within one period, during which each link is actuated. This signaling pattern may be found in,* e.g.*, a four-way signaling scheme in which only one link is actuated at any given time.*

Now suppose that links $\mathcal{L}^{pri} = \{\ell_1, \ell_2\}$ are actuated simultaneously during the primary *phase, and then links $\mathcal{L}^{sec} = \{\ell_3, \ell_4\}$ are actuated simultaneously during the* secondary *phase as in Fig.2(c). This is a commonly encountered situation in which the signal consists of two phases for the two directions of travel. Without loss of generality, we assume that $\Gamma_{\ell_1} = \Gamma_{\ell_2} = 0$. An important observation for this common case is that, regardless of the length of the green splits, we have $\Gamma_{\ell_3} = \Gamma_{\ell_4} = T/2$, and thus $\gamma_{\ell_3} = \gamma_{\ell_4} = \pi$.*

For $\ell, k \in \mathcal{L}$, let the *split ratio* $\beta_{\ell k}$ denote the fraction of vehicles that are routed to link k upon exiting link ℓ. We assume the split ratios are given and fixed, a standard modeling assumption since split ratios can be inferred from measured traffic flow [12]. Clearly, $\beta_{\ell k} \neq 0$ only if $\sigma(\ell) = \tau(k)$, and

$$\sum_{k \in \mathcal{L}} \beta_{\ell k} \leq 1 \quad \forall \ell \in \mathcal{L} \tag{3}$$

where strict inequality in (3) implies that a fraction of the vehicles departing link ℓ are routed off the network via an unmodeled road. We remark that our formulation is general enough to accommodate, *e.g.*, turn pocket lanes by introducing additional links for these lanes.

IV. Sinusoidal Approximation of Arrivals, Departures, and Queues

To address the signal coordination problem discussed above, we assume the arrival and departure processes satisfy a periodicity assumption:

Assumption 2 (Periodicity Assumption)**.** *The network is in periodic steady state so that all arrivals, departures, and queues are periodic with period T.*

When the green splits and turn ratios are able to accommodate the exogenous arrival rates, a natural condition, the network converges to a periodic steady state and thus the Periodicity Assumption is reasonable [13]. Under the Periodicity Assumption, we propose approximating the arrival and departure processes at a queue as sinusoids of appropriate amplitude and phase shift, an idea we formalize next.

A. Entry Links

For each $\ell \in \mathcal{E}$, we assume that the arrival of vehicles to the queue on link ℓ at signal $\sigma(\ell)$ is approximated by the function

$$\hat{a}_\ell(t) = A_\ell + \alpha_\ell \cos(\omega t - \varphi_\ell) \tag{4}$$

for constants $A_\ell, \alpha_\ell, \varphi_\ell \geq 0$ with $A_\ell \geq \alpha_\ell$. The constant A_ℓ represents the average arrival rate of vehicles to the queue on link ℓ; α_ℓ allows for periodic fluctuation in the arrival rate; $\Phi_\ell = \varphi_\ell/\omega$ is the offset of the periodic arrival of vehicles to the queue on link ℓ and includes the travel time of link ℓ. Throughout, we employ a *vertical queueing* model [14] such that travel time of a link is constant and does not change with queue length.

For $A_\ell = \alpha_\ell$, $\hat{a}_\ell(t)$ approximates a periodic pulse arrival, denoted by $a_\ell(t)$ as in Fig. 3(a), which is often used to model the arrival of vehicle platoons. If the periodic pulse has amplitude h_ℓ and duration δ_ℓ, we take $A_\ell = h_\ell\delta_\ell/T$ so that $\int_T \hat{a}_\ell(t)dt = \int_T a_\ell(t)dt = h_\ell\delta_\ell$. In this case, Φ_ℓ is the midpoint of the arrival pulse.

The departure process $d_\ell(t)$ for entry link $\ell \in \mathcal{E}$ is determined by the signal at intersection $\sigma(\ell) \in \mathcal{S}$ and must satisfy

$$\int_T d_\ell(t)\ dt = \int_T a_\ell(t)\ dt \tag{5}$$

by the Periodicity Assumption. We approximate the departure process as a shifted sinusoid

$$\hat{d}_\ell(t) = A_\ell\big(1 + \cos(\omega t - (\theta_{\sigma(\ell)} + \gamma_\ell))\big) \quad \forall \ell \in \mathcal{E}, \tag{6}$$

where, as defined above, $(\theta_{\sigma(\ell)} + \gamma_\ell)$ denotes the actuation offset of link ℓ as determined by the phase offset of signal $\sigma(\ell)$ and the green split of link ℓ; see Figure 3(b). We assume that all links have adequate capacity for upstream traffic and thus do not consider blocking of traffic by downstream congestion.

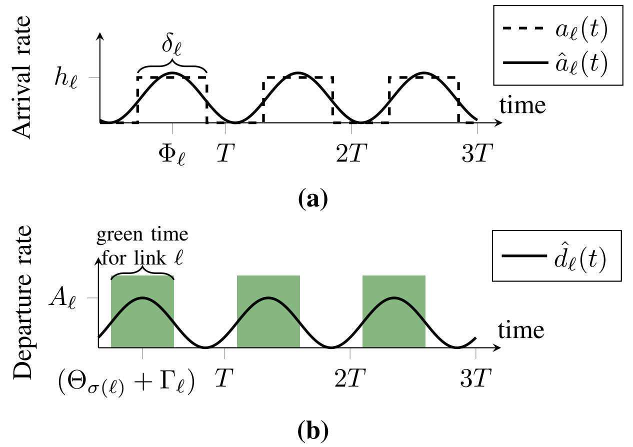

Fig. 3. Sinusoidal approximations of arrivals and departures. **(a)** A particular approximation of the arrival process on an entry link. The actual arrival process for link $\ell \in \mathcal{E}$ is a pulse train, which is approximated with a sinusoid. **(b)** The sinusoidal approximation of the departure process at any link $\ell \in \mathcal{L}$. The amplitude is required to match A_ℓ by the Periodicity Assumption, and the phase offset $\Theta_{\sigma(\ell)}$ is a design parameter.

B. Nonentry Links

For each link $\ell \in \mathcal{L}\backslash\mathcal{E}$, the arrival process is defined as follows. Let Λ_ℓ denote the travel time, in seconds, of link ℓ, and let $\lambda_\ell = \omega\Lambda_\ell$. The approximate arrival process for the queue on link $\ell \in \mathcal{L}\backslash\mathcal{E}$ is then

$$\begin{aligned}\hat{a}_\ell(t) &= \sum_{k\in\mathcal{L}} \beta_{k\ell}A_k(1 + \cos(\omega t - (\theta_{\sigma(k)} + \gamma_k) - \lambda_\ell)) &(7)\\ &= A_\ell + \alpha_\ell\cos(\omega t - (\theta_{\tau(\ell)} + \varphi_\ell)) &(8)\end{aligned}$$

where A_ℓ, α_ℓ, and φ_ℓ are given by the following formulas:

$$A_\ell = \sum_{k\in\mathcal{L}} \beta_{\ell k}A_k \tag{9}$$

$$\alpha_\ell^2 = \left(\sum_{k\in\mathcal{L}} \beta_{k\ell}A_k\cos(\gamma_k)\right)^2 + \left(\sum_{k\in\mathcal{L}} \beta_{k\ell}A_k\sin(\gamma_k)\right)^2 \tag{10}$$

$$\varphi_\ell = \lambda_\ell + \arctan\left(\frac{\sum_{k\in\mathcal{L}}\beta_{k\ell}A_k\sin(\gamma_k)}{\sum_{k\in\mathcal{L}}\beta_{k\ell}A_k\cos(\gamma_k)}\right). \tag{11}$$

In (7)–(8), we use the fact that $\sigma(k) = \tau(\ell)$ for any k, ℓ such that $\beta_{k\ell} \neq 0$. The approximate departure process is as described above for entry links, that is,

$$\hat{d}_\ell(t) = A_\ell\big(1 + \cos(\omega t - (\theta_{\sigma(\ell)} + \gamma_\ell))\big), \quad \forall \ell \in \mathcal{L}\backslash\mathcal{E}. \tag{12}$$

Note that A_ℓ, α_ℓ, λ_ℓ and φ_ℓ do not depend on the offsets $\{\theta_k\}_{k\in\mathcal{L}}$.

C. Queueing Process

Let $\hat{q}(t)$ denote the approximate queueing process that results from the sinusoidal approximation of arrivals and departures. Then

$$\begin{aligned}\dot{\hat{q}}_\ell(t) =& \hat{a}_\ell(t) - \hat{d}_\ell(t) &(13)\\ =& \alpha_\ell\cos(\omega t - (\theta_{\tau(\ell)} + \varphi_\ell))\\ &- A_\ell\cos(\omega t - (\theta_{\sigma(\ell)} + \gamma_\ell)) &(14)\\ =& Q_\ell\cos(\omega t - \xi_\ell) &(15)\end{aligned}$$

where

$$Q_\ell = \sqrt{A_\ell^2 + \alpha_\ell^2 - 2A_\ell\alpha_\ell \cos((\theta_{\tau(\ell)} + \varphi_\ell) - (\theta_{\sigma(\ell)} + \gamma_\ell))} \tag{16}$$

$$\xi_\ell = \arctan\left(\frac{\alpha_\ell \sin(\theta_{\tau(\ell)} + \varphi_\ell) - A_\ell \sin(\theta_{\sigma(\ell)} + \gamma_\ell)}{\alpha_\ell \cos(\theta_{\tau(\ell)} + \varphi_\ell) - A_\ell \cos(\theta_{\sigma(\ell)} + \gamma_\ell)}\right). \tag{17}$$

Thus, ξ_ℓ represents the normalized offset of the queueing process and $\Xi_\ell = \xi_\ell/\omega$ is the offset of the queueing process in seconds. It follows by integration that

$$\hat{q}_\ell(t) = \frac{Q_\ell}{\omega}\sin(\omega t - \xi_\ell) + C_\ell \tag{18}$$

where C_ℓ is the average queue length on link ℓ. Since $\hat{q}_\ell(t) \geq 0$ for all time, we have

$$C_\ell \geq (Q_\ell/\omega). \tag{19}$$

Under the assumption that all queues are emptied in each cycle, which is the case if the demand is *strictly feasible* [12], [13], then (19) holds with equality.

D. Performance Metric

For link $\ell \in \mathcal{L}$, consider the performance metric

$$J_\ell \triangleq (Q_\ell/\omega)^2. \tag{20}$$

When (19) holds with equality, J_ℓ is an approximate measure of the squared average queue length on link ℓ and (up to a constant factor) the maximum squared queue length on link ℓ. Note that A_ℓ, α_ℓ are functions only of the network topology, split ratios, green splits, and $\{A_\ell\}_{\ell\in\mathcal{E}}$, all of which are assumed fixed. Thus, minimizing J_ℓ is equivalent to maximizing the normalized reward function

$$R_\ell = A_\ell\alpha_\ell \cos((\theta_{\tau(\ell)} + \varphi_\ell) - (\theta_{\sigma(\ell)} + \gamma_\ell)). \tag{21}$$

The interpretation of maximizing R_ℓ is clear: for each link, we wish to minimize the difference between the arrival offset $(\theta_{\tau(\ell)} + \varphi_\ell)$ and the departure offset $(\theta_{\sigma(\ell)} + \gamma_\ell)$. Choosing

$$\theta_{\sigma(\ell)} + \gamma_\ell = \theta_{\tau(\ell)} + \varphi_\ell \tag{22}$$

optimizes the offset of signals $\sigma(\ell)$ and $\tau(\ell)$ from the perspective of reducing the queue length on link ℓ; the difficulty arises in choosing the offsets for all signals simultaneously, as it is generally not possible to satisfy (22) for all $\ell \in \mathcal{L}$. Efficiently computing an optimal tradeoff is addressed in the next section.

V. Optimizing Traffic Signal Offsets

A. Offset optimization problem and a convex relaxation

We propose $\sum_{\ell\in\mathcal{L}} R_\ell$ as the global objective function for the network and thus pose the following by substituting (21):

Offset Optimization Problem:

$$\underset{\{\theta_s\}_{s\in\mathcal{S}}}{\text{maximize}} \quad \sum_{\ell\in\mathcal{L}} A_\ell\alpha_\ell \cos(\theta_{\tau(\ell)} - \theta_{\sigma(\ell)} + \varphi_\ell - \gamma_\ell) \tag{23}$$

where it is understood that $\theta_s \in [0, 2\pi]$ for each $s \in \mathcal{S}$. In Sections III and IV, we have assumed $\theta_\epsilon = 0$. Here, we will include θ_ϵ as a decision variable in the optimization problem for convenience; this is clearly acceptable since the objective function in (23) is invariant to a shift of all θ_s, $s \in \mathcal{S} \cup \epsilon$. In particular, given a solution to (23), we may shift all phases by the nonzero θ_ϵ to obtain an alternative solution as is done below in (43)–(44).

In general, the optimization problem (23) is not convex. Here, we propose a semidefinite relaxation for approximately solving (23). We first recast (23) to an equivalent quadratically constrained quadratic program that is amenable to convex relaxation. For $s, u \in \mathcal{S}$, define[1]

$$\mathcal{L}_{s\to u} = \{\ell \in \mathcal{L} \mid \tau(\ell) = s \text{ and } \sigma(\ell) = u\}. \tag{24}$$

Let $W_i \in \mathbb{R}^{(|\mathcal{S}|+1)\times(|\mathcal{S}|+1)}$ for $i \in \{1, 2\}$ be given by

$$W_1[s, u] = \sum_{\ell\in\mathcal{L}_{s\to u}} A_\ell\alpha_\ell \cos(\varphi_\ell - \gamma_\ell) \tag{25}$$

$$W_2[s, u] = \sum_{\ell\in\mathcal{L}_{s\to u}} A_\ell\alpha_\ell \sin(\varphi_\ell - \gamma_\ell) \tag{26}$$

and let

$$\underline{W} = \begin{bmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{bmatrix}, \quad W = \frac{1}{2}(\underline{W} + \underline{W}^T). \tag{27}$$

Define the following:

$$x_s = \cos\theta_s \quad \forall s \in \mathcal{S} \cup \epsilon, \qquad x = \{x_s\}_{s\in\mathcal{S}\cup\epsilon} \tag{28}$$

$$y_s = \sin\theta_s \quad \forall s \in \mathcal{S} \cup \epsilon, \qquad y = \{y_s\}_{s\in\mathcal{S}\cup\epsilon} \tag{29}$$

$$z = (x, y). \tag{30}$$

By enumerating $\mathcal{S} \cup \epsilon$, we interpret $z \in \mathbb{R}^{2|\mathcal{S}|+2}$. We rewrite the objective function in (23):

$$\sum_{\ell\in\mathcal{L}} A_\ell\alpha_\ell \cos(\theta_{\tau(\ell)} - \theta_{\sigma(\ell)} + \varphi_\ell - \gamma_\ell) = z^T W z. \tag{31}$$

Furthermore, we have the constraint $x_s^2 + y_s^2 = 1$ for all $s \in \mathcal{S} \cup \epsilon$. Let $E_s \in \mathbb{R}^{|\mathcal{S}|+1}$ for $s \in \mathcal{S} \cup \epsilon$ be given by

$$E_s[u, v] = \begin{cases} 1 & \text{if } u = v = s \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

and define

$$M_s = \begin{bmatrix} E_s & 0 \\ 0 & E_s \end{bmatrix}. \tag{33}$$

Then the constraint $x_s^2 + y_s^2 = 1$ is equivalent to $z^T M_s z = 1$.

We formulate the following quadratically constrained quadratic program (QCQP), which is equivalent to the optimization problem (23):

Equivalent QCQP Formulation:

$$\underset{z\in\mathbb{R}^{2|\mathcal{S}|+2}}{\text{maximize}} \quad z^T W z \tag{34}$$

$$\text{subject to} \quad z^T M_s z = 1 \quad \forall s \in \mathcal{S} \cup \epsilon. \tag{35}$$

Given a solution z^* to (34)–(35), we may easily recover the corresponding phase offsets θ_s, $s \in \mathcal{S}$ by partitioning $z^* = (x^*, y^*)$ for which $\theta_s = \arctan(y_s^*/x_s^*)$. Since W is

[1] Note that $\mathcal{L}_{s\to u}$ may have cardinality greater than one if, *e.g.*, we model turn pocket lanes with additional links as remarked above.

not positive semidefinite and we have equality constraints in (35), the equivalent QCQP formulation above is not convex.

Let $Z = zz^T$, and recall that $z^T Y z = \mathbf{Tr}(z^T Y z) = \mathbf{Tr}(YZ)$ for any square matrix Y. By optimizing over a rank-one matrix Z instead of over a vector z, we equivalently write (34)–(35) as the following rank-constrained semidefinite program (SDP):

Equivalent Rank-Constrained SDP Formulation:

$$\underset{Z \in \mathbb{R}^{(2|\mathcal{S}|+2)\times(2|\mathcal{S}|+2)}}{\text{maximize}} \quad \mathbf{Tr}(WZ) \tag{36}$$

$$\text{subject to} \quad \mathbf{Tr}(M_s Z) = 1 \quad \forall s \in \mathcal{S} \cup \epsilon \tag{37}$$

$$Z \succeq 0 \tag{38}$$

$$\mathbf{rank}(Z) = 1 \tag{39}$$

Note that $\mathbf{Tr}(M_s Z) = 1$ is equivalent to the condition $Z[i,i] + Z[j,j] = 1$ for $i = 1, \ldots, |\mathcal{S}| + 1$ and $j = i + |\mathcal{S}| + 1$. Furthermore, the rank constraint (39) is nonconvex, rendering the equivalent rank-constrained SDP formulation a nonconvex optimization problem. However, we may obtain a convex relaxation by omitting this constraint:

Relaxed Convex SDP Formulation:

$$\underset{Z \in \mathbb{R}^{(2|\mathcal{S}|+2)\times(2|\mathcal{S}|+2)}}{\text{maximize}} \quad \mathbf{Tr}(WZ) \tag{40}$$

$$\text{subject to} \quad \mathbf{Tr}(M_s Z) = 1 \quad \forall s \in \mathcal{S} \cup \epsilon \tag{41}$$

$$Z \succeq 0. \tag{42}$$

The feasible set defined by (41)–(42) contains the feasible set defined by (37)–(39), and thus a solution to (40)–(42) provides an *upper bound* on the achievable performance of the optimization problem (23).

Interestingly, it has been observed that, for many practically motivated problems, solutions Z^* to the relaxed optimization problem are such that $\mathbf{rank}(Z^*) = 1$ and thus the true optimal solution z^* to the Offset Optimization Problem is obtained by factoring $Z^* = (z^*)(z^*)^T$, as is the case in the example below. Understanding when the relaxation returns the exactly optimal solution has been the subject of recent research [9], [15].

For instances when $\mathbf{rank}(Z^*) > 1$, we must approximate a solution to the Offset Optimization Problem from Z^*, for which a number of approaches exist [6]. For example, let $U \Sigma U^T = Z^*$ be an eigenvalue decomposition of Z^* where Σ is a diagonal matrix with the (nonnegative) eigenvalues of Z^* along the diagonal, and U is an orthonormal matrix with columns U_i, $i = 1, \ldots, 2|\mathcal{S}| + 2$. Assume Σ is ordered so that $\Sigma[1,1]$ is the largest eigenvalue of Z^*. Then we take $\hat{z} = U_1$ and partition $\hat{z} = (\hat{x}, \hat{y})$. We may obtain an estimate of the optimal offsets as follows:

$$\bar{\theta}_s \triangleq \arctan\left(\frac{\hat{y}_s}{\hat{x}_s}\right) \quad \forall s \in \mathcal{S} \cup \epsilon \tag{43}$$

$$\hat{\theta}_s = \bar{\theta}_s - \bar{\theta}_\epsilon \quad \forall s \in \mathcal{S}. \tag{44}$$

Other techniques exist for recovering a feasible solution from Z^*. In particular, [16], [17] propose randomized algorithms that use the Cholesky factorization of Z^* and provide bounds (in expectation) on the suboptimality of the obtained feasible solution.

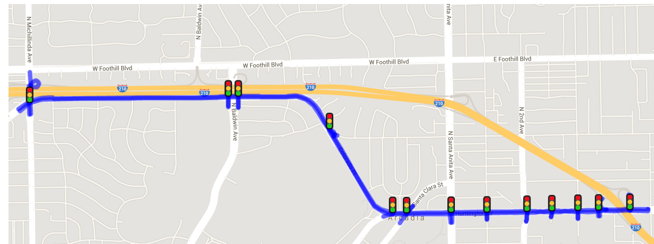

Fig. 4. Huntington Dr./Colorado Blvd. arterial corridor in Arcadia, California. The case study consists of 50 links and 13 signalized intersections.

Remark 1. *If there exists $s \in \mathcal{S} \cup \epsilon$ such that $W_1[s,u] = W_2[s,u] = 0$ for all $u \in \mathcal{S}$, then θ_s does not appear in the objective of* (23) *and the decision variables x_s and y_s should be removed from the optimization problem prior to the convex relaxation. For example, if $\alpha_\ell = 0$ for all $\ell \in \mathcal{E}$, as is in the case study where vehicles are assumed to arrive at entry links as a steady stream, then x_ϵ and y_ϵ should be removed from the optimization problem.*

VI. Case Study

We consider the Huntington Dr./Colorado Blvd. arterial corridor in Arcadia, California shown in Fig. 4 that is adjacent to Interstate 210. The network comprises 50 links and 13 signalized intersections. Traffic demand and empirical turn ratios were obtained from several sources as part of the Connected Corridors project at UC Berkeley [11].

The signalized intersections operate with fixed-time policies and hence with fixed green splits. In actuality, 11 of the 13 signalized intersections possess a common cycle time of 120 seconds; the remaining two intersections operate with cycle times of 90 seconds and 145 seconds. Our analysis requires a common cycle time for all intersections, and thus we have scaled the fixed green splits for these two intersections so that all signalized intersections operate with a common cycle time of 120 seconds for our study. We refer to this scenario, where phase offsets are obtained from given phase plans, as the *baseline* scenario in Table II.

Twenty-seven of the 50 links constitute the two-way arterial along Huntington Dr. and Colorado Blvd. to the entry ramp of I-210 along Michillinda Avenue. The remaining links direct exogenous traffic onto the network at intersections and constitute $\mathcal{E}$. We assume vehicles arrive as a steady stream to all links $\ell \in \mathcal{E}$, that is, $\alpha_\ell = 0$ for all $\ell \in \mathcal{E}$. This implies $w_\ell = 0$ for all $\ell \in \mathcal{E}$, and thus the queues on all entry links do not appear in the optimization formulation (this reflects the fact that the phase offset policy only shifts, in time, the queue length process on entry links and does not affect the peak or average queues on these links). This further implies that θ_ϵ, the phase offset of the exogenous signals, does not appear in the objective function of the optimization procedure and is thus omitted as noted in Remark 1.

We employ the convex, relaxed optimization problem (40)–(42) using the MATLAB-based CVX package [18],

Metric		Baseline	Optimized	Improvement
Mean of Peak Queue Lengths[a,b]	[veh]	17.14	12.47	27.2%
Mean of Avg. Queue Lengths[b]	[veh]	5.12	3.72	27.4%
Avg. Time in Network	[s]	127.9	116.6	8.9%

[a]To account for stochastic fluctuations in the simulation, the peak queue is defined to be the queue length that is surpassed by the queue process only one percent of the time.

[b]The summation is over $\mathcal{L}\backslash\mathcal{E}$ as noted in Section VI.

TABLE II
CASE STUDY RESULTS

which returns an optimal Z^* in 0.37 seconds on a standard laptop. For this problem, CVX returned Z^* such that $\mathbf{rank}(Z^*) = 1$, that is, the approximate solution is in fact a global solution to the problem. This is to be expected since the underlying graph structure is acyclic for which the relaxation has been shown to be exact [15]; when determining the "underlying graph structure", we only consider the existence of a link between a pair of signals and not the direction of travel. In Table II, we refer to the scenario where phase offsets are the optimized phase offsets as the *optimized* scenario.

To evaluate the performance of the optimized phase offsets, we simulate the network using the PointQ simulator [19], a mesoscopic simulation environment developed at UC Berkeley. The simulator is event driven and models individual vehicles in the network which travel along links, join finite capacity queues at intersections, and execute turn movements (which are randomly realized according to fixed turn ratios) when queues are actuated. The interarrival time for vehicles at entry links is exponentially distributed to approximate the steady stream of arriving vehicles. The simulation is run for a sufficiently long time period so that stochastic fluctuations are negligible.

Results of this simulation are shown in Table II. We see an improvement of approximately 27% for both the mean peak queue lengths and the mean average queue lengths on links $\mathcal{L}\backslash\mathcal{E}$ (recall that entry links are excluded from the optimization procedure since vehicles join these links according to a constant exogenous arrival process).

Reducing queue lengths is the explicit performance metric employed in this paper and results in a number of direct benefits (*e.g.*, reduced emissions and likelihood of accidents due to reduced speed oscillations, reduced "wasted green time" caused by downstream congestion that blocks flow). Reduced queue lengths also afford proximate improvement in other metrics. For example, in this simulation, the offset optimization procedure results in decreased travel time of nearly 9% for all vehicles in the network. Future research will further investigate the connection between offset optimization and additional metrics, particularly as it relates to the "bandwidth" maximization problem [2].

VII. CONCLUSIONS

We have developed a scalable approximation algorithm to the offset optimization problem in arbitrary network topologies. Despite the simplifying assumptions in the problem formulation, the case study based on real traffic data demonstrated a reduction in the queue lengths and travel time. Future work will further investigate under what conditions the semidefinite relaxation is exact. In addition, a weighted objective function may be used to, *e.g.*, encourage short queues on short links to prevent congestion and is another direction for future research.

VIII. ACKNOWLEDGEMENTS

The authors thank John Maidens for suggesting the connection between (23) and rank relaxations of QCQPs.

REFERENCES

[1] J. Little, M. Kelson, and N. Gartner, "MAXBAND: A versatile program for setting signals on arteries and triangular networks," *Transportation Research Record*, vol. 795, 1981.

[2] G. Gomes, "Bandwidth maximization using vehicle arrival functions," *IEEE Transactions on Intelligent Transportation Systems*, vol. 16, pp. 1977–1988, Aug 2015.

[3] N. Gartner and J. Little, "The generalized combination method for area traffic control," *Transportation Research Record*, pp. 58–69, 1975.

[4] N. Gartner and C. Stamatiadis, "Arterial-based control of traffic flow in urban grid networks," *Mathematical and Computer Modelling*, vol. 35, pp. 657–671, 2002.

[5] N. Gartner and C. Stamatiadis, "Progression optimization featuring arterial and route based priority signal networks," *Journal of Intelligent Transportation Systems: Technology, Planning, and Operations*, vol. 8:2, pp. 77–86, 2004.

[6] Z.-Q. Luo, W.-K. Ma, A.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," *Signal Processing Magazine, IEEE*, vol. 27, no. 3, pp. 20–34, 2010.

[7] S. Bose, D. Gayme, K. M. Chandy, and S. H. Low, "Quadratically constrained quadratic programs on acyclic graphs with application to power flow." `http://arxiv.org/abs/1203.5599`, 2012.

[8] S. Low, "Convex relaxation of optimal power flow part I: Formulations and equivalence," *IEEE Transactions on Control of Network Systems*, vol. 1, pp. 15–27, March 2014.

[9] A. S. Bandeira, N. Boumal, and A. Singer, "Tightness of the maximum likelihood semidefinite relaxation for angular synchronization." `http://arxiv.org/abs/1411.3272`, 2014.

[10] A. Singer, "Angular synchronization by eigenvectors and semidefinite programming," *Applied and Computational Harmonic Analysis*, vol. 30, pp. 20–36, 2011.

[11] "Analysis, modeling, and simulation workshop at Caltrans D7." `www.connected-corridors.berkeley.edu/analysis-modeling-and-simulation-workshop-caltrans-d7-0`.

[12] P. Varaiya, "The max-pressure controller for arbitrary networks of signalized intersections," in *Advances in Dynamic Network Modeling in Complex Transportation Systems*, pp. 27–66, Springer, 2013.

[13] A. Muralidharan, R. Pedarsani, and P. Varaiya, "Analysis of fixed-time control," *Transportation Research Part B: Methodological*, vol. 73, pp. 81–90, 2015.

[14] P. Varaiya, "Max pressure control of a network of signalized intersections," *Transportation Research Part C: Emerging Technologies*, vol. 36, pp. 177–195, 2013.

[15] S. Sojoudi and J. Lavaei, "Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure," *SIAM Journal on Optimization*, vol. 24, no. 4, pp. 1746–1778, 2014.

[16] A. Nemirovski, C. Roos, and T. Terlaky, "On maximization of quadratic form over intersection of ellipsoids with common center," *Mathematical Programming*, vol. 86, no. 3, pp. 463–473, 1999.

[17] A. M.-C. So, J. Zhang, and Y. Ye, "On approximating complex quadratic optimization problems via semidefinite programming relaxations," *Mathematical Programming*, vol. 110, no. 1, pp. 93–110, 2007.

[18] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1." `http://cvxr.com/cvx`, Mar. 2014.

[19] J. Lioris, A. Kurzhanskiy, D. Triantafyllos, and P. Varaiya, "Control experiments for a network of signalized intersections using the '.Q'," *12th IFAC-IEEE Workshop on Discrete Event Systems, Ecole Normale Superieure de Cachan, France*, 2014.