A Contractive Approach to Separable Lyapunov Functions for Monotone Systems

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Abstract

Monotone systems preserve a partial ordering of states along system trajectories and are often amenable to separable Lyapunov functions that are either the sum or the maximum of a collection of functions of a scalar argument. In this paper, we consider constructing separable Lyapunov functions for monotone systems that are also contractive, that is, the distance between any pair of trajectories exponentially decreases. The distance is defined in terms of a possibly state-dependent norm. When this norm is a weighted one-norm, we obtain conditions which lead to sum-separable Lyapunov functions, and when this norm is a weighted infinity-norm, symmetric conditions lead to max-separable Lyapunov functions. In addition, we consider two classes of Lyapunov functions: the first class is separable along the system's state, and the second class is separable along components of the system's vector field. The latter case is advantageous for many practically motivated systems for which it is difficult to measure the system's state but easier to measure the system's velocity or rate of change. In addition, we present an algorithm based on sum-of-squares programming to compute such separable Lyapunov functions. We provide several examples to demonstrate our results.

1 Introduction

A dynamical system is *monotone* if it maintains a partial ordering of states along trajectories of the system [18, 19, 42]. Monotone systems exhibit structure and ordered behavior that is exploited for analysis and control, *e.g.*, [1, 2, 11]. Monotone systems theory has been applied to biological networks [45], transportation networks [9, 17, 26], and large-scale and distributed control applications [15, 38].

It has been observed that monotone systems are often amenable to separable Lyapunov functions for stability analysis. In particular, classes of monotone systems have been identified that allow for Lyapunov functions that are the sum or maximum of a collection of functions of a scalar argument [15, 39, 49]. In the case of linear monotone systems, also called *positive* systems, such sum-separable and max-separable global Lyapunov functions are always possible when the origin is a stable equilibrium [38]. For nonlinear monotone systems with a stable equilibrium, It is shown in [15] that max-separable Lyapunov functions can always be constructed in compact subsets of the domain of attraction, and it is shown in [49] that such max-separable Lyapunov functions can be obtained from the leading eigenfunction of the linear, but infinite dimensional, Koopman operator associated with the monotone system. Also in [15], small-gain type formulas are provided that give sufficient conditions for constructing sum-separable Lyapunov functions of planar monotone systems. Moreover, [15] provides counterexamples showing that a stable monotone system may allow for both, either, or neither type of separable Lyapunov functions globally [15].

A dynamical system is *contractive* if the distance between states along any pair of trajectories is exponentially decreasing [16, 24, 36, 46]. When an equilibrium exists, contraction implies global convergence and a Lyapunov function is given by the distance to the equilibrium. The magnitude of the vector field provides an alternative Lyapunov function. Certain classes of monotone systems have been shown to be also contractive with respect to non-Euclidean norms. For example, [29, 31, 40] study a model for gene translation which is monotone and contractive with respect to a weighted ℓ_1 norm. A closely related result is obtained for transportation flow networks in [6, 8]. In [8], a Lyapunov function defined as the magnitude of the vector field is used, and in [6], a Lyapunov function based on the distance of the state to the equilibrium is used.

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In this paper, we establish sufficient conditions for constructing sum-separable and max-separable Lyapunov functions for monotone systems by appealing to contraction theory. In particular, we study monotone systems that are contractive with respect to a possibly state-dependent, weighted ℓ_1 norm, which leads to sum-separable Lyapunov functions, or weighted ℓ_∞ norm, which leads to max-separable Lyapunov functions. We first provide sufficient conditions establishing contraction for monotone systems in terms of negativity of scaled row or column sums of the Jacobian matrix for the system. The scaling may be state-dependent, and our results rely on the theory of Finsler-Lyapunov functions [16]. By allowing for Lyapunov functions derived from state-dependent norms, we significantly extend our prior work [7], which only considered constant norms over the state-space. In addition, the recent work [28] seeks to determine when a contracting monotone system has a separable contraction metric. Unlike the present work, which focuses on norms that are naturally separable, namely, the ℓ_1 and ℓ_∞ norm, [28] considers Riemannian metrics and studies when a contracting system also has a separable Riemannian contraction metric.

In addition to deriving Lyapunov functions that are separable along the state of the system, we also introduce Lyapunov functions that are separable along components of the vector field. This is especially relevant for certain classes of systems such as multiagent control systems or flow networks where it is often more practical to measure velocity or flow rather than position or state. Additionally, we present results of independent interest for proving asymptotic stability and obtaining Lyapunov functions of systems that are *nonexpansive* with respect to a particular vector norm, *i.e.*, the distance between states along any pair of trajectories does not increase. Finally, we draw connections between our results and related results, particularly small-gain theorems for interconnected input-to-state stable (ISS) systems.

This paper is organized as follows. Section 2 defines notation and Section 3 provides the problem setup. Section 4 contains the statements of our main results. Before proving these results, we review contraction theory for general, potentially time-varying nonlinear systems in Section 5. In particular, we establish explicit characterizations of contraction in terms of the matrix measure of the Jacobian matrix for the system dynamics, allowing for potentially state-dependent norms by appealing to Finsler-Lyapunov theory [16]. In applications, it is often the case that the system dynamics are not quite contractive, but are nonexpansive with respect to a particular norm, *i.e.*, the distance between any pair of trajectories does not increase for all time. In Section 6, we provide a sufficient condition for establishing global asymptotic stability for nonexpansive systems.

In Section 7, we provide the proofs of our main results, and in Section 8, we use our main results to establish a numerical algorithm for searching for separable Lyapunov functions using sum-of-squares programming. We provide several applications of our results in Section 9. Next, we discuss how the present work relates to results for interconnected ISS systems and generalized contraction theory in Section 10. Section 11 contains concluding remarks.

2 Preliminaries

Throughout, all inequalities are interpreted elementwise. The vector of all ones is denoted by $\mathbf{1}$. For functions of one variable, we denote derivative with the prime notation $'$. The ℓ_1 and ℓ_∞ norms are denote by $|\cdot|_1$ and $|\cdot|_\infty$, respectively, that is, $|x|_1 = \sum_{i=1}^n |x_i|$ and $|x|_\infty = \max_{i=1,\dots,n} |x_i|$ for $x \in \mathbb{R}^n$.

Let $|\cdot|$ be some vector norm on $\mathbb{R}^n$ and let $\|\cdot\|$ be its induced matrix norm on $\mathbb{R}^{n\times n}$. The corresponding *matrix measure* of the matrix $A \in \mathbb{R}^{n\times n}$ is defined as (see, *e.g.*, [14])

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h}. \tag{1}$$

One useful property of the matrix measure is that $\mu(A) < 0$ implies A is Hurwitz [32]. To see this, let λ be an eigenvalue of A with eigenvector v, and assume $|v| = 1$. Then $\|I + hA\| \geq |1 + h\lambda||v| = |1 + h\lambda|$. Moreover, $\mathrm{Re}(\lambda) = \lim_{h\to 0^+} \frac{1}{h}(|1 + h\lambda| - 1) \leq \lim_{h\to 0^+} \frac{1}{h}(\|I + hA\| - 1) = \mu(A)$ so that $\mathrm{Re}(\lambda) \leq \mu(A)$.

For the ℓ_1 norm, the induced matrix measure is given by

$$\mu_1(A) = \max_{j=1,\dots,n} \left(A_{jj} + \textstyle\sum_{i\neq j} |A_{ij}| \right) \tag{2}$$

for any $A \in \mathbb{R}^{n\times n}$. Likewise, for the ℓ_∞ norm, the induced matrix measure is given by

$$\mu_\infty(A) = \max_{i=1,\dots,n}\left(A_{ii} + \textstyle\sum_{j\neq i}|A_{ij}|\right). \tag{3}$$

See, *e.g.*, [14, Section II.8, Theorem 24], for a derivation of the induced matrix measures for common vector norms.

A matrix $A \in \mathbb{R}^{n\times n}$ is *Metzler* if all of its off diagonal components are nonnegative, that is, $A_{ij} \geq 0$ for all $i \neq j$. When A is Metzler, (2) and (3) reduce to

$$\mu_1(A) = \max_{j=1,\dots,n}\textstyle\sum_{i=1}^n A_{ij}, \tag{4}$$
$$\mu_\infty(A) = \max_{i=1,\dots,n}\textstyle\sum_{j=1}^n A_{ij}, \tag{5}$$

that is, $\mu_1(A)$ is the largest column sum of A and $\mu_\infty(A)$ is the largest row sum of A.

3 Problem Setup

Consider the dynamical system

$$\dot{x} = f(x) \tag{6}$$

for $x \in \mathcal{X} \subseteq \mathbb{R}^n$ where $f(\cdot)$ is continuously differentiable. Let $f_i(x)$ indicate the ith component of f and let $J(x) = \frac{\partial f}{\partial x}(x)$ be the Jacobian matrix of f.

Denote by $\phi(t, x_0)$ the solution to (6) at time t when the system is initialized with state x_0 at time 0. We assume that (6) is forward complete and $\mathcal{X}$ is forward invariant for (6) so that $\phi(t, x_0) \in \mathcal{X}$ for all $t \geq 0$ and all $x_0 \in \mathcal{X}$.

Except in Sections 5 and 6, we assume (6) is monotone [1, 42]:

Definition 1. *The system* (6) *is* monotone *if the dynamics maintain a partial order on solutions, that is,*

$$x_0 \leq y_0 \implies \phi(t, x_0) \leq \phi(t, y_0) \quad \forall t \geq 0 \tag{7}$$

for any $x_0, y_0 \in \mathcal{X}$.

In this paper, monotonicity is defined with respect to the positive orthant since inequalities are interpreted componentwise, although it is common to consider monotonicity with respect to other cones [1].

The following proposition characterizes monotonicity using the Jacobian matrix $J(x)$.

Proposition 1 (*Kamke Condition*, [42, Ch. 3.1])**.** *Assume $\mathcal{X}$ is convex. Then the system* (6) *is monotone if and only if the Jacobian $J(x)$ is Metzler for all $x \in \mathcal{X}$.*

Here, we are interested in certifying stability of an equilibrium x^* for the dynamics (6). To that end, we have the following definition of Lyapunov function, the existence of which implies asymptotic stability of x^*.

Definition 2 ([44, p. 219])**.** *Let x^* be an equilibrium of* (6). *A continuous function $V : \mathcal{X} \to \mathbb{R}$ is a* (local) Lyapunov function *for* (6) *with respect to x^* if on some neighborhood $\mathcal{O}$ of x^* the following hold:*

1. *$V(x)$ is proper at x^*, that is, for small enough $\epsilon > 0$, $\{x \in \mathcal{X} | V(x) \leq \epsilon\}$ is a compact subset of $\mathcal{O}$;*
2. *$V(x)$ is positive definite on $\mathcal{O}$, that is, $V(x) \geq 0$ for all $x \in \mathcal{O}$ and $V(x) = 0$ if and only if $x = x^*$;*
3. *For any $x_0 \in \mathcal{O}$, $x_0 \neq x^*$, there is some time $\tau > 0$ such that $V(\phi(\tau, x_0)) < V(x_0)$ and $V(\phi(t, x_0)) \leq V(x_0)$ for all $t \in (0, \tau]$.*

Furthermore, V is a global Lyapunov function for (6) *if we may take $\mathcal{O} = \mathcal{X}$ and $V(x)$ is globally proper, that is, for each $L > 0$, $\{x \in \mathcal{X} | V(x) \leq L\}$ is compact.*

In this paper, we are particularly interested in Lyapunov functions defined using nondifferentiable norms, thus we rely on the definition above which only requires $V(x)$ to be continuous. Such nondifferentiable Lyapunov functions will not pose additional technical challenges since we will not rely on direct computation of the gradient of $V(x)$. Instead, we will construct locally Lipschitz continuous Lyapunov functions and bound the time derivative evaluated along trajectories of the system, which exists for almost all time. Nonetheless, classical Lyapunov theory, which verifies condition (3) of the above definition by requiring $(\partial V/\partial x)\cdot f(x) < 0$ for all x, is extended to such nondifferentiable Lyapunov functions with the use of generalized derivatives [5].

Note that when $\mathcal{X} = \mathbb{R}^n$, a Lyapunov function is globally proper (and hence a global Lyapunov function) if and only if it is radially unbounded [44, p. 220].

We call the Lyapunov function $V(x)$ *agent sum-separable* if it decomposes as

$$V(x) = \sum_{i=1}^{n} V_i(x_i, f_i(x)) \tag{8}$$

for a collection of functions V_i, and *agent max-separable* if it decomposes as

$$V(x) = \max_{i=1,\dots,n} V_i(x_i, f_i(x)). \tag{9}$$

If each V_i in (8) (respectively, (9)) is a function only of x_i, we further call $V(x)$ *state sum-separable* (respectively, *state max-separable*). On the other hand, if each V_i is a function only of $f_i(x)$, we say $V(x)$ is *flow state-separable* (respectively, *flow max-separable*).

Our objective is to construct separable Lyapunov functions for a class of monotone systems.

4 Main Results

In the main result of this paper, we provide conditions for certifying that a monotone system possesses a globally asymptotically stable equilibrium. These conditions then lead to easily constructed separable Lyapunov functions. As discussed in Section 5, this result relies on establishing that the monotone system is contractive with respect to a possibly state-dependent norm. We will first state our main result in this section before developing the theory required for its proof in the sequel.

Definition 3. *The set $\mathcal{X} \subseteq \mathbb{R}^n$ is* rectangular *if there exists a collection of connected sets (*i.e., *intervals) $\mathcal{X}_i \subseteq \mathbb{R}$ for $i = 1, \dots, n$ such that $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$.*

Theorem 1. *Let (6) be a monotone system with rectangular domain $\mathcal{X}$. Consider a collection of C^1 functions $\theta_i : \mathcal{X}_i \to \mathbb{R}$ for $i = 1, \dots, n$ and define $\theta(x) := \begin{bmatrix}\theta_1(x_1) & \theta_2(x_2) & \dots & \theta_n(x_n)\end{bmatrix}^T$. Suppose there exists $c_1 > 0$ such that $\theta(x) \geq c_1\mathbf{1}$ for all $x \in \mathcal{X}$ and the following conditions hold:*

1. *For all $x \in \mathcal{X}$,*

$$\theta(x)^T J(x) + \dot{\theta}(x)^T \leq 0. \tag{10}$$

2. *There exists a positively invariant set $\mathcal{K} \subseteq \mathcal{X}$ and constant $c_2 > 0$ such that, for all $x \in \mathcal{K}$,*

$$\theta(x)^T J(x) + \dot{\theta}(x)^T \leq -c_2\theta(x)^T. \tag{11}$$

Then there exists a globally asymptotically stable equilibrium $x^* \in \mathcal{K}$. Furthermore,

$$V(x) = \sum_{i=1}^{n} \left| \int_{x_i^*}^{x_i} \theta_i(\sigma) d\sigma \right| \tag{12}$$

is a global Lyapunov function, and

$$V(x) = \sum_{i=1}^{n} \theta_i(x_i) |f_i(x_i)| \tag{13}$$

is a local Lyapunov function. If (13) is globally proper, then it is also a global Lyapunov function.

Above, $\dot{\theta}(x)$ is shorthand for

$$\dot{\theta}(x) = \frac{\partial \theta}{\partial x} f(x) = \begin{bmatrix}\theta_1'(x_1)f_1(x) & \dots & \theta_n'(x_n)f_n(x)\end{bmatrix}^T. \tag{14}$$

Theorem 2. *Let (6) be a monotone system with rectangular domain $\mathcal{X}$. Consider a collection of C^1 functions $\omega_i : \mathcal{X}_i \to \mathbb{R}$ for $i = 1, \dots, n$ and define $\omega(x) = \begin{bmatrix}\omega_1(x_1) & \omega_2(x_2) & \dots & \omega_n(x_n)\end{bmatrix}$. Suppose $\omega(x) > 0$ for all $x \in \mathcal{X}$, there exists $c_1 > 0$ such that $\omega(x) \leq c_1 \mathbf{1}^T$ for all $x \in \mathcal{X}$, and the following conditions hold:*

1. *For all $x \in \mathcal{X}$,*

$$J(x)\omega(x) - \dot{\omega}(x) \leq 0. \tag{15}$$

2. *There exists a positively invariant set $\mathcal{K} \subseteq \mathcal{X}$ and constant $c_2 > 0$ such that, for all $x \in \mathcal{K}$,*

$$J(x)\omega(x) - \dot{\omega}(x) \leq -c_2\omega(x). \tag{16}$$

Then there exists a globally asymptotically stable equilibrium $x^* \in \mathcal{K}$. Furthermore,

$$V(x) = \max_{i=1,\dots,n} \left| \int_{x_i^*}^{x_i} \frac{1}{\omega_i(\sigma)} d\sigma \right| \tag{17}$$

is a global Lyapunov function and

$$V(x) = \max_{i=1,\dots,n} \frac{1}{\omega_i(x_i)} |f_i(x_i)| \tag{18}$$

is a local Lyapunov function. If (18) is globally proper, then it is also a global Lyapunov function.

Note that (12) and (17) are state-separable Lyapunov functions, while (13) and (18) are agent-separable Lyapunov functions.

If x^* is known *a priori*, then condition (2) of Theorems 1 and 2 may be replaced with an easily checkable condition on $J(x^*)$. In particular, we may replace condition (2) in Theorem 1 with the requirement that

$$\theta(x^*)^T J(x^*) < 0. \tag{19}$$

Likewise, condition 2 of Theorem 2 may be replaced with

$$J(x^*)\omega(x^*) < 0 \tag{20}$$

when x^* is known. This observation is made precise in the following Proposition.

Proposition 2. *Let x^* be an equilibrium of the monotone system $\dot{x} = f(x)$ and consider C^1 function $\theta : \mathcal{X} \to \mathbb{R}^n$ satisfying $\theta(x^*) > 0$. If $\theta(x^*)^T J(x^*) < 0$, then condition (2) of Theorem 1 holds.*

Likewise, for C^1 function $\omega : \mathcal{X} \to \mathbb{R}^n$ satisfying $\omega(x^*) > 0$, if $J(x^*)\omega(x^*) < 0$, then condition (2) of Theorem 2 holds.

Proof. Suppose the equilibrium x^* is such that $\theta(x^*)^T J(x^*) < 0$. Note that $\dot{\theta}(x^*) = 0$ since $f(x^*) = 0$ so that there exists $c_2 > 0$ satisfying $\theta(x^*)^T J(x^*) + \dot{\theta}(x^*) < -c_2\theta(x^*)^T$. By continuity of θ and $\dot{\theta}$, there exists an open neighborhood of x^* for which (11) holds.

It remains to show that this neighborhood can be further restricted to be positively invariant. Let $\Theta^* = \text{diag}\{\theta(x^*)\}$. Then $\theta(x^*)^T J(x^*) = \mathbf{1}^T \Theta^* J(x^*)$ so that $\mathbf{1}^T \Theta^* J(x^*) < 0$, which implies $\mathbf{1}^T \Theta^* J(x^*)(\Theta^*)^{-1} < 0$, that is, each column sum of $\tilde{J}^* := \Theta^* J(x^*)(\Theta^*)^{-1}$ is strictly negative. Moreover, $\tilde{J}^*$ is Metzler since $J(x^*)$ is Metzler, and it follows that $\mu_1(\tilde{J}^*) < 0$ by (4) so that $\tilde{J}^*$ is Hurwitz. The matrix $J(x^*)$ is related to $\tilde{J}^*$ via a similarity transform so that also $J(x^*)$ is Hurwitz and thus x^* is exponentially stable. It follows that there exists a positively invariant neighborhood of x^*. Combining these observations, there exists a positively invariant neighborhood of x^* for which (11) holds.

When $J(x^*)\omega(x^*) < 0$, a similar argument as above establishes a constant c_2 and open neighborhood of x^* for which (16) holds. Let $\Omega^* = \text{diag}\{\omega(x^*)\}$ and define $\tilde{J}^* := (\Omega^*)^{-1} J(x^*)\Omega^*$, for which a parallel argument as above shows $\tilde{J}^*\mathbf{1} < 0$, that is, each row sum of $\tilde{J}^*$ is strictly negative. Since J^* is again Metzler, it follows from (5) that $\mu_\infty(\tilde{J}^*) < 0$ so that $\tilde{J}^*$ and also $J(x^*)$ are Hurwitz, and thus there exists a positively invariant neighborhood of x^* for which (16) holds. □

Example 1. *Consider the system*

$$\dot{x}_1 = -x_1 + x_2^2 \tag{21}$$

$$\dot{x}_2 = -x_2 \tag{22}$$

which is monotone on the invariant domain $\mathcal{X} = (\mathbb{R}_{\geq 0})^2$ *with unique equilibrium* $x^* = (0, 0)$*. The Jacobian is given by*

$$J(x) = \begin{bmatrix} -1 & 2x_2 \\ 0 & -1 \end{bmatrix}. \tag{23}$$

Let $\theta(x) = \begin{bmatrix} 1 & x_2 + 1 \end{bmatrix}^T$ *so that* $\dot{\theta}(x) = \begin{bmatrix} 0 & -x_2 \end{bmatrix}^T$ *and*

$$\theta(x)^T J(x) + \dot{\theta}(x)^T = \begin{bmatrix} -1 & -1 \end{bmatrix} \leq 0. \tag{24}$$

Since also $\theta(x^*)^T J(x^*) = \begin{bmatrix} -1 & -1 \end{bmatrix}$*, Theorem 1 is applicable, where we substitute* (19) *for condition (2) of the theorem via Proposition 2. Therefore,* $x^* = 0$ *is globally asymptotically stable and, from* (12) *and* (13)*, either of the following are global Lyapunov functions:*

$$V(x) = x_1 + x_2 + \frac{1}{2}x_2^2 \tag{25}$$

$$V(x) = \left|-x_1 + x_2^2\right| + x_2 + x_2^2. \tag{26}$$

On the other hand, let $\omega(x) = \begin{bmatrix} 2 & \frac{1}{1+x_2} \end{bmatrix}^T$ *for which* $\dot{\omega}(x) = \begin{bmatrix} 0 & \frac{x_2}{(1+x_2)^2} \end{bmatrix}^T$ *and*

$$J(x)\omega(x) - \dot{\omega}(x) = \begin{bmatrix} -2 + \frac{2x_2}{1+x_2} \\ \frac{-1}{1+x_2} - \frac{x_2}{(1+x_2)^2} \end{bmatrix} \leq 0 \tag{27}$$

for all $x \in \mathcal{X}$*. Since also* $J(x^*)\omega(x^*) = \begin{bmatrix} -2 & -1 \end{bmatrix}^T$*, Theorem 2 is applicable, where we substitute* (20) *for condition (2) of the theorem via Proposition 2. Thus, from* (17) *and* (18)*, either of the following are also Lyapunov functions:*

$$V(x) = \max\left\{\frac{1}{2}x_1, x_2 + \frac{1}{2}x_2^2\right\} \tag{28}$$

$$V(x) = \max\left\{\frac{1}{2}\left|-x_1 + x_2^2\right|, x_2 + x_2^2\right\}. \tag{29}$$

■

The following example shows how the local Lyapunov function (13) may fail to be a global Lyapunov function when it is not globally proper.

Example 2. *Consider the scalar system* $\dot{x} = f(x) = e^{-x} - 1$ *evolving in* $\mathcal{X} = \{x : x \geq 0\}$*, for which* $J(x) = -e^{-x} \leq 0$ *for all* $x \in \mathcal{X}$ *and* $J(0) < 0$*, and let* $\theta(x) \equiv 1$*. Then Theorem 1 implies* $V(x) = |x|$ *is a global Lyapunov function and* $V(x) = |f(x)|$ *is a local Lyapunov function, however,* $|f(x)|$ *is not a global Lyapunov function since it is not globally proper.*

Remark 1. *Even when* (13) *or* (18) *is not globally proper and thus not a global Lyapunov function, as in the example above, it is nonetheless the case that both functions monotonically decrease to zero along any trajectory of the system.*

We now specialize Theorems 1 and 2 to the case where $\theta(x)$ and $\omega(x)$ are independent of x, that is, are constant vectors. This special case proves to be especially useful in a number of applications as demonstrated in Section 9.

Corollary 1. *Let* (6) *be a monotone system with equilibrium* x^**. Suppose there exists a vector* $v > 0$ *such that the following conditions hold:*

1. $v^T J(x) \le 0$ *for all* $x \in \mathcal{X}$,

2. *There exists a positively invariant set* $\mathcal{K} \subseteq \mathcal{X}$ *and constant* $c > 0$ *such that*

$$v^T J(x) \le -c\mathbf{1}^T \qquad \text{for all } x \in \mathcal{K}. \tag{30}$$

Then there exists a globally asymptotically stable equilibrium $x^* \in \mathcal{K}$. *Furthermore,*

$$V(x) = \sum_{i=1}^{n} v_i |x_i - x_i^*| \tag{31}$$

is a global Lyapunov function and

$$V(x) = \sum_{i=1}^{n} v_i |f_i(x)| \tag{32}$$

is a local Lyapunov function. If condition 2 above holds with $\mathcal{K} = \mathcal{X}$, *then* (32) *is also a global Lyapunov function.*

Corollary 2. *Let* (6) *be a monotone system with equilibrium* x^*. *Suppose there exists a vector* $w > 0$ *such that the following conditions hold:*

1. $J(x)w \le 0$ *for all* $x \in \mathcal{X}$,

2. *There exists a positively invariant set* $\mathcal{K} \subseteq \mathcal{X}$ *and constant* $c > 0$ *such that*

$$J(x)w \le -c\mathbf{1} \qquad \text{for all } x \in \mathcal{K}. \tag{33}$$

Then there exists a globally asymptotically stable equilibrium $x^* \in \mathcal{K}$. *Furthermore,*

$$V(x) = \max_{i=1,\dots,n} \left\{ \frac{1}{w_i} |x_i - x_i^*| \right\} \tag{34}$$

is a global Lyapunov function and

$$V(x) = \max_{i=1,\dots,n} \left\{ \frac{1}{w_i} |f_i(x)| \right\} \tag{35}$$

is a local Lyapunov function. If condition 2 above holds with $\mathcal{K} = \mathcal{X}$, *then* (35) *is also a global Lyapunov function.*

Remark 2. *Mirroring Proposition 2, in the case that* x^* *is known, condition 2 in Corollary 1 (respectively, Corollary 2) is replaced with* $v^T J(x^*) < 0$ *(respectively,* $J(x^*)w < 0$*).*

Note that (31) and (34) are state separable Lyapunov functions while (32) and (35) are flow separable Lyapunov functions.

The following example shows that Corollaries 1 and 2 recover a well-known condition for stability of monotone linear systems.

Example 3 (Linear systems)**.** *Consider* $\dot{x} = Ax$ *for* A *Metzler. Corollaries 1 and 2 imply that if one of the following conditions holds, then the system is globally asymptotically stable:*

$$\text{There exists } v > 0 \text{ such that } v^T A < 0, \quad \text{or} \tag{36}$$

$$\text{There exists } w > 0 \text{ such that } Aw < 0. \tag{37}$$

If (36) *holds then* $V(x) = \sum_{i=1}^n v_i |x_i|$ *and* $V(x) = \sum_{i=1}^n v_i |(Ax)_i|$ *are Lyapunov functions, and if* (37) *holds then* $V(x) = \max_i \{|x_i|/w_i\}$ *and* $V(x) = \max_i \{|(Ax)_i|/w_i\}$ *are Lyapunov functions where* $(Ax)_i$ *denotes the* i*th element of* Ax. ■

In fact, it is well known that A is Hurwitz if and only if either (and, therefore, both) of the two conditions (36) and (37) hold, as noted in, *e.g.*, [38, Proposition 1], and the corresponding state separable Lyapunov functions of Example 3 are also derived in [38]. Thus, Theorems 1 and 2, along with Corollaries 1 and 2, may be considered nonlinear extensions of these results.

The proofs of Theorems 1 and 2 use contraction theoretic arguments and, specifically, show that a monotone system satisfying the hypotheses of the theorems is contractive with respect to a suitably defined, state-dependent norm. The proof technique illuminates useful properties of contractive systems that are of independent interest and appear to be novel. These results are presented next, before returning to the proofs of the above theorems.

5 Contraction with respective to state-dependent, non-Euclidean metrics

In the following two sections, we develop preliminary results necessary to prove our main results of Section 4. Here, we do not require the monotonicity property. Moreover, we develop our results for potentially time-varying systems, that is, systems of the form $\dot{x} = f(t, x)$.

We first provide conditions for establishing that a nonlinear system is *contractive*. A system is contractive with respect to a given metric if the distance between any two trajectories decreases at an exponential rate. Contraction with respect to Euclidean norms with potentially state-dependent weighting matrices is considered in [24]. This approach equips the state-space with a Riemannian structure. For metrics defined using non-Euclidean norms, which has proven useful in many applications, a Riemannian approach is insufficient, and contraction has been characterized using matrix measures for fixed (i.e., state-independent) norms [46]. These approaches were recently unified and generalized in [16] using the theory of Finsler-Lyapunov functions.

Here, we consider contraction with respect to potentially state-dependent, non-Euclidean norms. Our results are an application of the Finsler-Lyapunov theory of [16]. However, to the best of our knowledge, such explicit characterizations of contraction with respect to state-dependent, non-Euclidean norms using matrix measures are not established elsewhere in the literature.

Definition 4. *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ and for a connected set $\mathcal{X} \subseteq \mathbb{R}^n$, let $\Theta : \mathcal{X} \to \mathbb{R}^{n\times n}$ be continuously differentiable and uniformly positive definite, that is, $\Theta(x) \geq \alpha I$ for all $x \in \mathcal{X}$ for some $\alpha > 0$. For any two points $x, y \in \mathcal{X}$, let $\Gamma(x, y)$ be the set of piecewise C^1 curves $\gamma : [0, 1] \to \mathcal{X}$ connecting x to y so that $\gamma(0) = x$ and $\gamma(1) = y$. The induced* distance metric *is given by*

$$d(x, y) = \inf_{\gamma\in\Gamma(x,y)} \int_0^1 |\Theta(\gamma(s))\gamma'(s)|ds. \tag{38}$$

Remark 3. *From a differential geometric perspective, the function $V(x, \delta x) := |\Theta(x)\delta x|$ is a* Finsler function *defined on the tangent bundle of $\mathcal{X}$, and the distance metric $d(x, y)$ is the* Finsler metric *associated with F [3].*

Note that, because $\Theta(x)$ is uniformly positive definite, there exists a constant $m > 0$ such that $|\Theta(x)z| \geq m|z|$ for any x, z. It follows that $d(x, y) \geq \inf_{\gamma\in\Gamma(x,y)} \int_0^1 m|\gamma'(s)|ds = m|y - x|$.

In the remainder of this section and in the following section, we study the system

$$\dot{x} = f(t, x) \tag{39}$$

for $x \in \mathcal{X} \subseteq \mathbb{R}^n$ where $f(t, x)$ is differentiable in x, and $f(t, x)$ and the Jacobian $J(t, x) \triangleq \frac{\partial f}{\partial x}(t, x)$ are continuous in (t, x). As before, $\phi(t, x_0)$ denotes the solution to (39) at time t when the system is initialized with state x_0 at time $t = 0$.

Proposition 3. *Consider the system* (39) *and suppose $\mathcal{K} \subseteq \mathcal{X}$ is forward invariant and connected. Let $|\cdot|$ be a norm on $\mathbb{R}^n$ with induced matrix measure $\mu(\cdot)$ and let $\Theta : \mathcal{K} \to \mathbb{R}^{n\times n}$ be continuously differentiable and* *uniformly positive definite so that there exists $\alpha > 0$ for which $\Theta(x) \geq \alpha I$ for all $x \in \mathcal{K}$. If there exists $c \geq 0$ such that*

$$\mu\left(\frac{\partial\Theta}{\partial x}(x)f(t,x)\Theta(x)^{-1} + \Theta(x)J(t,x)\Theta(x)^{-1}\right) \leq -c \tag{40}$$

for all $x \in \mathcal{K}$ and all $t \geq 0$, then

$$d(\phi(t,y_0), \phi(t,x_0)) \leq e^{-ct}d(y_0,x_0) \tag{41}$$

for all $x_0, y_0 \in \mathcal{K}$. Moreover, if the system is time-invariant so that $\dot{x} = f(x)$, then

$$|\Theta(\phi(t,x_0))f(\phi(t,x_0))| \leq e^{-ct}|\Theta(x_0)f(x_0)| \tag{42}$$

for all $x_0 \in \mathcal{K}$ and all $t \geq 0$.

Before proceeding with the proof of Proposition 3, we first note that when $\Theta(x) \equiv \hat{\Theta} > 0$ for some fixed $\hat{\Theta} \in \mathbb{R}^{n\times n}$, then $|\cdot|_{\hat{\Theta}}$ defined as $|z|_{\hat{\Theta}} = |\hat{\Theta}z|$ for all z is a norm and $d(x,y) = |y-x|_{\hat{\Theta}}$. The corresponding induced matrix measure satisfies $\mu_{\hat{\Theta}}(A) = \mu(\hat{\Theta}A\hat{\Theta}^{-1})$ for all A so that Proposition 3 states: if $\mu_{\hat{\Theta}}(J(t,x)) \leq -c < 0$ for all t and x, then $|\phi(t,x_0) - \phi(t,y_0)|_{\hat{\Theta}} \leq e^{-ct}|x_0 - y_0|_{\hat{\Theta}}$ and, in the time-invariant case, $|f(\phi(t,x_0))|_{\hat{\Theta}} \leq e^{-ct}|f(x_0)|_{\hat{\Theta}}$, which recovers familiar results for contractive systems with respect to (state-independent) non-Euclidean norms; see, *e.g.*, [46, Theorem 1], [7, Proposition 2]. Thus, Proposition 3 is an extension of these results to state-dependent, non-Euclidean norms.

Proof. Let $V(x,\delta x) = |\Theta(x)\delta x|$. To prove (41), we claim that $V(x,\delta x)$ is a Lyapunov function for the *variational* dynamics

$$\dot{x} = f(t,x) \tag{43}$$

$$\dot{\delta x} = J(t,x)\delta x. \tag{44}$$

In particular,

$$\dot{V}(x,\delta x) \leq -cV(x,\delta x) \tag{45}$$

along trajectories of the variational system. Recall from Remark 3 that $V(x,\delta x)$ is a Finsler function defined on the tangent bundle of $\mathcal{X}$. Assuming the claim to be true, it follows that $V(x,\delta x)$ is then a *Finsler-Lyapunov* function as defined in [16] for $\dot{x} = f(t,x)$ where δx is the *virtual displacement* associated with the system. It is proved in [16, Theorem 1] that (45) implies (41) for all $x_0, y_0 \in \mathcal{K}$ where $d(x,y)$ is the Finsler distance as defined in (38).

To prove the claim, let $(x(t), \delta x(t))$ be some trajectory of the variational dynamics (43)–(44). In the following, we omit dependent variables from the notation when clear and write, *e.g.*, f, J, and Θ instead of $f(t,x)$, $J(t,x)$, and $\Theta(x)$. We then have

$$\dot{V}(x(t),\delta x(t)) \tag{46}$$

$$:= \lim_{h\to 0^+}\frac{V(x(t+h),\delta x(t+h)) - V(x(t),\delta x(t))}{h} \tag{47}$$

$$= \lim_{h\to 0^+}\frac{V(x+hf,\delta x+hJ\delta x) - V(x,\delta x)}{h} \tag{48}$$

$$= \lim_{h\to 0^+}\frac{|\Theta(x+hf)(\delta x+hJ\delta x)| - |\Theta(x)\delta x|}{h} \tag{49}$$

$$= \lim_{h\to 0^+}\frac{|(\Theta+h\frac{\partial\Theta}{\partial x}f)(\delta x+hJ\delta x)| - |\Theta\delta x|}{h} \tag{50}$$

$$= \lim_{h\to 0^+}\frac{|\Theta\delta x+h\Theta J\Theta^{-1}\Theta\delta x+h\frac{\partial\Theta}{\partial x}f\Theta^{-1}\Theta\delta x| - |\Theta\delta x|}{h} \tag{51}$$

$$\leq \lim_{h\to 0^+}\frac{|I+h(\Theta J\Theta^{-1}+\frac{\partial\Theta}{\partial x}f\Theta^{-1})||\Theta\delta x| - |\Theta\delta x|}{h} \tag{52}$$

$$= \mu\left(\Theta J\Theta^{-1} + \frac{\partial\Theta}{\partial x}f\Theta^{-1}\right)|\Theta\delta x| \tag{53}$$

$$\leq -c|\Theta(x)\delta x|, \tag{54}$$

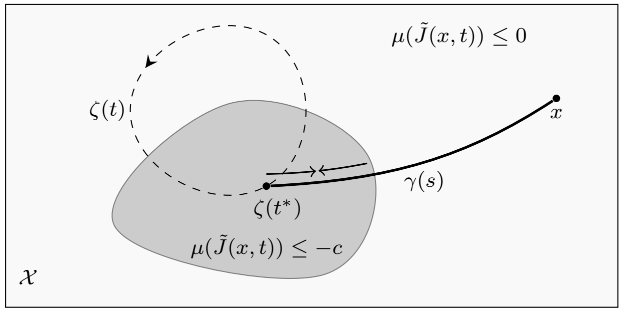

Figure 1: For a periodic, nonexpansive system with domain $\mathcal{X}$, $\mu(\tilde{J}(x,t)) \leq 0$ for all x, t. If there exists a periodic trajectory $\xi(t)$ and a time t^* such that $\mu(\tilde{J}(t^*, \zeta(t^*))) < 0$, then there exists a neighborhood of $\zeta(t^*)$ and an interval of time during which the distance between any other trajectory and the periodic trajectory strictly decreases. It follows that all trajectories must entrain to the periodic trajectory.

and we have proved (45) so that (41) follows by [16, Theorem 1].

To prove (42), now suppose $\dot{x} = f(x)$. Then $\dot{f}(x) = J(x)f(x)$ so that $(x(t), f(x(t)))$ is a trajectory of the variational system for any trajectory $x(t)$ of $\dot{x} = f(x)$. Equation (42) then follows immediately from (45). □

We note that (42) can be proved directly by applying a change of coordinates. In particular, for $\dot{x} = f(x)$ and $\Theta(x)$ as in the hypotheses of Proposition 3, consider the time-varying change of variables $w := \Theta(x)f(x)$ for which $\dot{w} = \tilde{J}(x)w$ where

$$\tilde{J}(x) := \frac{\partial \Theta}{\partial x}(x)f(x)\Theta^{-1}(x) + \Theta(x)J(x)\Theta^{-1}(x). \tag{55}$$

By Coppel's Lemma (see, *e.g.*, [14, Theorem II.8.27]), $|w(t)| \leq e^{\int_0^t \mu(\tilde{J}(x(t))}|w(0)| \leq e^{-ct}|w(0)|$ where the second inequality follows from (40), providing an alternative interpretation for (42).

Definition 5. *A system for which the hypotheses of Proposition 3 hold with $c < 0$ (respectively, $c = 0$) is* contractive *(respectively,* nonexpansive*) with respect to $|\cdot|$ and $\Theta(x)$ on $\mathcal{K}$.*

Corollary 3. *Suppose the system* (39) *is contractive on $\mathcal{K}$ and time-invariant. Then there exists an asymptotically stable equilibrium point $x^* \in \mathcal{K}$, and the region of attraction includes $\mathcal{K}$ so that x^* is the unique equilibrium within $\mathcal{K}$.*

Proof. Since $\Theta(x)$ is uniformly positive definitive, there exists m such that $|\Theta(\phi(t,x_0))f(\phi(t,x_0))| \geq m|f(\phi(t,x_0))|$ for all $t \geq 0$. It follows from (42) that $|f(\phi(t,x_0))|$ satisfies an exponential bound in t for all $x_0 \in \mathcal{K}$, and thus $\phi(t,x_0)$ tends towards a finite equilibrium point for all $x_0 \in \mathcal{K}$. Since $\mathcal{K}$ cannot have more than one equilibrium point, all trajectories tend towards a unique $x^* \in \mathcal{K}$. □

6 A global asymptotic stability result for nonexpansive systems

When the hypotheses of Proposition 3 are only satisfied with $c = 0$, the system is nonexpansive as defined above so that the distance between any pair of trajectories is nonincreasing but not necessarily exponentially decreasing as when the system is contractive, *i.e.*, when $c < 0$. Nonetheless, if the vector field is periodic and there exists a periodic trajectory that passes through a region in which the contraction property holds locally, then all trajectories entrain, that is, converge, to this periodic trajectory. As a special case, if there exists an open set around an equilibrium in which the contraction property holds locally, then the equilibrium is globally asymptotically stable. We make these statements precise in this section.

Theorem 3. *Consider $\dot{x} = f(t,x)$ for $x \in \mathcal{X} \subseteq \mathbb{R}^n$ where $f(t,x)$ is differentiable in x, and $f(t,x)$ and the Jacobian $J(t,x) \triangleq \frac{\partial f}{\partial x}(t,x)$ are continuous in (t,x). Assume $\mathcal{X}$ is forward invariant and connected. Let $|\cdot|$*

be a norm on $\mathbb{R}^n$ with induced matrix measure $\mu(\cdot)$ and let $\Theta : \mathcal{X} \to \mathbb{R}^{n\times n}$ be continuously differentiable and uniformly positive definite so that there exists α for which $\Theta(x) \geq \alpha I$ for all $x \in \mathcal{X}$.

Suppose $f(t,x)$ is T-periodic for some $T > 0$ so that $f(t,x) = f(t+T,x)$ for all t and all $x \in \mathcal{X}$. Let $\zeta(t)$ be a periodic trajectory of the system and define

$$\tilde{J}(t,x) := \frac{\partial \Theta}{\partial x}(x) f(t,x)\Theta(x)^{-1} + \Theta(x)J(t,x)\Theta(x)^{-1}. \tag{56}$$

If $\mu(\tilde{J}(t,x)) \leq 0$ for all $x \in \mathcal{X}$ and $t \geq 0$, and there exists a time t^* such that

$$\mu(\tilde{J}(t^*, \zeta(t^*))) < 0, \tag{57}$$

then all trajectories entrain to $\zeta(t)$, that is,

$$\lim_{t\to\infty} d(\phi(t,x_0), \zeta(t)) = 0 \quad \textit{for all } x_0 \in \mathcal{X}. \tag{58}$$

Proof. Without loss of generality, assume $t^* = 0$. Note that $\Theta(x)^{-1}$ is continuous since $\Theta(x)$ is continuous and $\Theta(x)$ is uniformly positive definite. Then, condition (57) and continuity of $J(t,x)$, $\Theta(x)^{-1}$, and $(\partial\Theta/\partial x)(x)$ imply there exists $\epsilon > 0$, $c > 0$, and $0 < \tau \leq T$ such that

$$\mu(\tilde{J}(t,y)) \leq -c \qquad \forall t \in [0,\tau],\ \forall y \in B_\epsilon(\zeta(t)) \tag{59}$$

where $\mathcal{B}_\epsilon(y) = \{z : d(y,z) \leq \epsilon\}$. Define the mapping

$$P(\xi) = \phi(T,\xi) \tag{60}$$

and observe that $P^k(\xi) = \phi(kT,\xi)$. Let $\zeta^* = \zeta(0)$ and note that ζ^* is a fixed point of P. Consider a point $\xi \in \mathcal{B}_\epsilon(\zeta^*)$. Let $x(t) = \phi(t,\xi)$ and note that $d(\zeta^*, P(\xi)) = d(\zeta(T), x(T))$. We have $d(\zeta(T),x(T)) \leq d(\zeta(\tau),x(\tau))$ since $\mu(\tilde{J}(t,x)) \leq 0$ for all $x \in \mathcal{X}$ and $t \geq 0$, and, following the proof of Proposition 3, (59) implies

$$d(\zeta(\tau),x(\tau)) \leq e^{-c\tau} d(\zeta(0),x(0)). \tag{61}$$

Now consider $\xi \in \mathcal{X}$ such that $d(\zeta^*,\xi) > \epsilon$ and again let $x(t) = \phi(t,\xi)$. Let $\delta = (1-e^{-c\tau})\epsilon/2$ and let $\gamma(s) : [0,1] \to \mathcal{X}$ with $\gamma(0) = \zeta^*$ and $\gamma(1) = \xi$ be such that $\int_0^1 |\Theta(\gamma(s))\gamma'(s)|ds \leq d(\zeta^*,\xi) + \delta$. Since $d(\zeta^*,\gamma(0)) = 0$ and $d(\zeta^*,\gamma(1)) > \epsilon$, by continuity of $\gamma(s)$ and $d(\zeta^*,\cdot)$, there exists s_ϵ such that $d(\zeta^*,\gamma(s_\epsilon)) = \epsilon$. Moreover,

$$d(\zeta^*,\gamma(s_\epsilon)) + d(\gamma(s_\epsilon),\xi) \tag{62}$$

$$\leq \int_0^{s_\epsilon} |\Theta(\gamma(s))\gamma'(s)|ds + \int_{s_\epsilon}^1 |\Theta(\gamma(s))\gamma'(s)|ds \tag{63}$$

$$\leq d(\zeta^*,\xi) + \delta \tag{64}$$

where the first inequality follows from the definition of d and the fact that arc-length is independent of reparameterizations of γ.

Let $\sigma(t) = \phi(t,\gamma(s_\epsilon))$. By following the proof of Proposition 3 with $c = 0$, $\mu(\tilde{J}(t,x)) \leq 0$ for all $x \in \mathcal{X}$ and all $t \geq 0$ implies $d(\phi(t,x_0),\phi(t,y_0)) \leq d(x_0,y_0)$ for all $x_0,y_0 \in \mathcal{X}$. In particular, $d(\sigma(T),x(T)) \leq d(\sigma(0),x(0)) = d(\gamma(s_\epsilon),\xi) \leq d(\zeta^*,\xi) + \delta - \epsilon$. Furthermore, by the same argument as in the preceding case, we have $d(\zeta(T),\sigma(T)) \leq e^{-c\tau} d(\zeta^*,\gamma(s_\epsilon)) = e^{-c\tau}\epsilon$. Thus, by the triangle inequality,

$$d(\zeta^*,P(\xi)) \leq d(\zeta(T),\sigma(T)) + d(\sigma(T),x(T)) \tag{65}$$

$$\leq d(\zeta^*,\xi) + \delta - (1-e^{-c\tau})\epsilon \tag{66}$$

$$= d(\zeta^*,\xi) - \delta. \tag{67}$$

We thus have

$$d(\zeta^*,P(\xi)) \leq \begin{cases} d(\zeta^*,\xi) - \delta & \text{if } d(\zeta^*,\xi) > \epsilon \\ e^{-c\tau} d(\zeta^*,\xi) & \text{if } d(\zeta^*,\xi) \leq \epsilon. \end{cases} \tag{68}$$

It follows that, for all ξ, $d(\zeta^*,P^k(\xi)) \leq \epsilon$ for some finite k (in particular, for any $k \geq d(\zeta^*,\xi)/\delta$). The second condition of (68) ensures $d(\zeta^*,P^k(\xi)) \to 0$ as $k \to \infty$ so that (58) holds. ◻

Theorem 3 and its proof are illustrated in Figure 1.

Note that, as a consequence of the global entrainment property, $\gamma(t)$ in the statement of Theorem 3 is the unique periodic trajectory of a system satisfying the hypotheses of the theorem.

Theorem 3 is closely related to existing results in the literature, although we believe the generality provided by Theorem 3 is novel. In particular, [26, Lemma 6] provides a similar result for $\mu(\cdot)$ restricted to the matrix measure induced by the ℓ_1 norm under the assumption that (6) is monotone and time-invariant. A similar technique is applied to periodic trajectories of a class of monotone flow networks in [25, Proposition 2], but a general formulation is not presented.

While Theorem 3 is interesting in its own right, in this paper, our main interest is in the following Corollary, which specializes Theorem 3 to time-invariant systems.

Corollary 4. *Consider $\dot{x} = f(x)$ for $x \in \mathcal{X} \subseteq \mathbb{R}^n$ for continuously differentiable $f(x)$. Assume $\mathcal{X}$ is forward invariant and connected. Let $|\cdot|$ be a norm on $\mathbb{R}^n$ with induced matrix measure $\mu(\cdot)$ and let $\Theta : \mathcal{X} \to \mathbb{R}^{n\times n}$ be continuously differentiable and uniformly positive definite so that there exists α for which $\Theta(x) \geq \alpha I$ for all $x \in \mathcal{X}$. Let x^* be an equilibrium of the system and define*

$$\tilde{J}(x) := \frac{\partial \Theta}{\partial x}(x) f(x) \Theta(x)^{-1} + \Theta(x) J(x) \Theta(x)^{-1} \tag{69}$$

where $J(x) := \frac{\partial f}{\partial x}(x)$. If

$$\mu(\tilde{J}(x)) \leq 0 \quad \text{for all } x \in \mathcal{X}, \text{ and} \tag{70}$$

$$\mu(\tilde{J}(x^*)) < 0, \tag{71}$$

then x^* is unique and globally asymptotically stable. Moreover,

$$V(x) = d(x, x^*) \tag{72}$$

is a global Lyapunov function and

$$V(x) = |\Theta(x) f(x)| \tag{73}$$

is a local Lyapunov function. If (73) is globally proper, then it is also a global Lyapunov function.

Proof. Choose any $T > 0$, for which $f(x)$ is then (vacuously) T-periodic, and $\zeta(t) := x^*$ is trivially a periodic trajectory so that Theorem 3 applies. Note that we may take $\tau = T$ in the proof of the theorem. From Proposition 3, with $\mathcal{K}$ taken to be $\mathcal{X}$ and $c = 0$, we have that $d(x, x^*)$ and $|\Theta(x) f(x)|$ are nonincreasing by (41) and (42), and both tend to zero along trajectories of the system since x^* is globally asymptotically stable, thus they both satisfy condition (3) of Definition 2 and therefore (72) and (73) are local Lyapunov functions. Since $d(x, x^*) \geq m|x^* - x|$ for some $m > 0$ for all x, $d(x, x^*)$ is globally proper so that (72) is a global Lyapunov function. □

7 Proof of main result

We are now in a position to prove our main results, Theorems 1 and 2. We begin with the following proposition, which shows that $d(x, y)$ as in (38) can be obtained explicitly when $\Theta(x)$ is a diagonal, state-dependent weighting matrix.

Proposition 4. *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a rectangular set and for any $x, y \in \mathcal{X}$, let $\Gamma(x, y)$ be the set of piecewise C^1 curves $\gamma : [0, 1] \to \mathcal{X}$ connecting x to y so that $\gamma(0) = x$ and $\gamma(1) = y$. Consider $\Theta : \mathcal{X} \to \mathbb{R}^{n\times n}$ given by $\Theta(x) = \mathrm{diag}\{\theta_1(x_1), \ldots, \theta_n(x_n)\}$ where $\{\theta_i(\cdot)\}_{i=1}^n$ is a collection of nonnegative C^1 functions. Suppose there exists $c > 0$ such that $\theta_i(x_i) > c$ for all $x \in \mathcal{X}$ for all $i = 1, \ldots, n$. Let $|\cdot|$ be a norm on $\mathbb{R}^n$ and consider the distance metric $d(x, y)$ given by (38). Then*

$$d(x, y) = \left| \begin{bmatrix} \int_{x_1}^{y_1} \theta_i(\sigma_1) d\sigma_1 & \ldots & \int_{x_n}^{y_n} \theta_i(\sigma_n) d\sigma_n \end{bmatrix}^T \right|. \tag{74}$$

Proof. Given any curve $\gamma \in \Gamma(x,y)$, define a new curve $\tilde{\gamma}(s) : [0,1] \to \mathbb{R}^n$ as follows. Let $\tilde{\gamma}(0) = x$ and let $\tilde{\gamma}_i'(s) = \theta_i(\gamma_i(s))\gamma_i'(s)$ for all i so that

$$\int_0^1 |\Theta(\gamma(s))\gamma'(s)|ds = \int_0^1 |\tilde{\gamma}'(s)|ds. \tag{75}$$

Observe that, for all i,

$$\tilde{\gamma}_i(1) - x_i = \int_0^1 \theta_i(\gamma_i(s))\gamma_i'(s)ds = \int_{x_i}^{y_i} \theta_i(\sigma)d\sigma. \tag{76}$$

Then $z := \tilde{\gamma}(1)$ is a point depending only on x and y and is independent of the particular curve γ. It follows that we may instead consider all curves between x and z. In particular,

$$\inf_{\tilde{\gamma}\in\Gamma(x,z)} \int_0^1 \left| \begin{bmatrix} \tilde{\gamma}_1'(s) & \dots & \tilde{\gamma}_n'(s) \end{bmatrix}^T \right| ds = |z - x|, \tag{77}$$

and, moreover, the infimum is achieved with the choice $\tilde{\gamma}(s) = (1-s)x + sz$ for which $\tilde{\gamma}'(s) = z - x$. For this choice of $\tilde{\gamma}$, there exists a unique curve $\gamma \in \Gamma(x,y)$ satisfying the decoupled system of differential equations $z_i - x_i = \theta_i(\gamma_i(s))\gamma_i'(s)$ for all i. Moreover, $\gamma_i(s)$ is monotonic for all i and all $s \in [0,1]$ since $\gamma_i'(s)$ does not change sign, and thus $\gamma(s)$ is contained in the rectangle defined by the corners x and y so that $\gamma(s) \in \mathcal{X}$ for $s \in [0,1]$. It follows that $d(x,y) = |z - x|$, which is equivalent to (74). □

Proof of Theorem 1. Let

$$\Theta(x) := \text{diag}\{\theta_1(x_1), \dots, \theta_n(x_n)\} \tag{78}$$

and consider the one-norm $|\cdot|_1$ with induced matrix measure $\mu_1(\cdot)$. Define $\tilde{J}(x)$ as in (69). Recall $J(x) := \frac{\partial f}{\partial x}(x)$ is Metzler since the system is monotone. Then also $\tilde{J}(x)$ is Metzler since $\Theta(x)$ is diagonal and positive definite so that $\Theta(x)J(x)\Theta(x)^{-1}$ retains the sign structure of $J(x)$ and $\frac{\partial \Theta}{\partial x}(x)f(x)\Theta(x)^{-1}$ is a diagonal matrix.

Then

$$\mu_1(\tilde{J}(x)) = \max_{j=1\dots,n} \left(\sum_{i=1}^n \tilde{J}_{ij}(x) \right) \tag{79}$$

$$= \max_{j=1\dots,n} \left(\left(\dot{\theta}_j(x) + \sum_{i=1}^n \theta_i(x)J_{ij}(x) \right) \theta_j(x)^{-1} \right) \tag{80}$$

where the first equality follows by (4) since $\tilde{J}(x)$ is Metzler. It follows from (10) and (80) that $\mu_1(\tilde{J}(x)) \le 0$ for all $x \in \mathcal{X}$, and from (11) and (80) that $\mu_1(\tilde{J}(x)) \le -c_2 < 0$ for all $x \in \mathcal{K}$. By Proposition 3 and Corollary 3, there exists equilibrium $x^* \in \mathcal{K}$, and, by (11), $\mu_1(\tilde{J}(x^*)) < 0$. Applying Corollary 4 establishes that (12) is a global Lyapunov function and (13) is a local Lyapunov function. □

Proof of Theorem 2. Let $\theta_i(x) := 1/\omega_i(x)$ and define

$$\Omega(x) := \text{diag}\{\omega_1(x_1), \dots, \omega_n(x_n)\} \tag{81}$$

$$\Theta(x) := \text{diag}\{\theta_1(x_1), \dots, \theta_n(x_n)\}. \tag{82}$$

Since $0 < \omega_i(x) \le c_1$ for all $x \in \mathcal{X}$ for $i = 1, \dots, n$, we have $0 < c_1^{-1} \le \theta_i(x)$ for all $x \in \mathcal{X}$ for $i = 1, \dots, n$.

Observe that

$$\dot{\theta}_i(x)\theta_i(x_i)^{-1} = \frac{\frac{d\theta_i}{dx_i}(x_i)}{\theta_i(x_i)} f_i(x) = \frac{-\frac{d\omega_i}{dx_i}(x_i)}{\omega_i(x_i)} f_i(x) \tag{83}$$

$$= -\dot{\omega}_i(x)\omega_i(x_i)^{-1} \tag{84}$$

for all $i = 1, \ldots, n$ where the second equality is established from the identity

$$\frac{d\omega_i}{dx_i}(x_i) = \frac{d}{dx_i}\left(\frac{1}{\theta_i(x_i)}\right) = \frac{-(d\theta_i/dx_i)(x_i)}{(\theta_i(x_i))^2}. \tag{85}$$

As in the proof of Theorem 2, define $\tilde{J}(x)$ according to (69), and now consider the infinity-norm $|\cdot|_\infty$ with induced matrix measure $\mu_\infty(\cdot)$. As before, $\tilde{J}(x)$ is Metzler so that

$$\mu_\infty(\tilde{J}(x)) = \max_{i=1,\ldots,n} \sum_{j=1}^{n} \tilde{J}_{ij}(x) \tag{86}$$

$$= \max_{i=1,\ldots,n} \left(\dot{\theta}_i(x)\theta_i(x)^{-1} + \theta_i(x_i)\sum_{j=1}^{n} J_{ij}(x)\theta_j(x_j)^{-1}\right) \tag{87}$$

$$= \max_{i=1,\ldots,n} \left(\omega_i(x_i)^{-1}\left(- \dot{\omega}_i(x) + \sum_{j=1}^{n} J_{ij}(x)\omega_j(x_j)\right)\right) \tag{88}$$

where the first equality follows from (5) and the last equality follows from (83)–(84). Then (15) and (88) imply that $\mu_\infty(\tilde{J}(x)) \leq 0$ for all $x \in \mathcal{X}$, and (16) and (88) imply that $\mu_\infty(\tilde{J}(x)) \leq -c_2$ for all $x \in \mathcal{K}$. By Proposition 3 and Corollary 3, there exists equilibrium $x^* \in \mathcal{K}$, and, by (16), $\mu_\infty(\tilde{J}(x^*)) < 0$. Applying Corollary 4 establishes that (17) is a global Lyapunov function and (18) is a local Lyapunov function. □

Proofs of Corollaries 1 and 2. We first show that condition 2 of Corollary 1 (respectively, Corollary 2) implies condition 2 of Theorem 1 (respectively, Theorem 2). To see this, suppose (30) holds and let $\theta(x) \equiv v$. Then $\dot{\theta}(x) \equiv 0$. Moreover, there exists some $\tilde{c} > 0$ such that $v^T J(x) \leq -\tilde{c}v$ for all $x \in K$, in particular, we take $\tilde{c} \in [c/|v|_\infty, 0)$. Therefore, (11) holds for all $x \in \mathcal{K}$ with c_2 taken to be $\tilde{c}$. A symmetric result holds for Corollary 2.

We now show that when condition 2 of either Corollary holds for $\mathcal{K} = \mathcal{X}$, then (13) (respectively, (18)) is a global Lyapunov function. To prove the claim, we need only show that (13) (respectively, (18)) is globally proper. We show that this is the case when (13) of Corollary 1 holds. The parallel result for Corollary 2 follows from a symmetric argument.

To this end, suppose (30) holds with $\mathcal{K} = \mathcal{X}$ for some $c > 0$. Let $\Theta = \text{diag}\{v\}$ so that (13) implies $\mu(\Theta J(x)\Theta^{-1}) \leq -\tilde{c}$ for all $x \in \mathcal{X}$ where $\tilde{c} > 0$ is as constructed above. Note that (13) is equivalent to $V(x) = |\Theta f(x)|_1$. From a slight modification of [14, Theorem 33, pp. 34–35], we have that $|\Theta f(x)|_1 \geq \tilde{c}|\Theta x|_1$, which implies that $V(x)$ is globally proper. For completeness, we repeat this argument here. Since $\Theta f(x) = \int_0^1 \Theta J(sx)x \, ds$,

$$|\Theta f(x)|_1 \geq -\mu_1\left(\int_0^1 \Theta J(sx)\Theta^{-1} \, ds\right)|\Theta x|_1 \tag{89}$$

$$\geq -\left(\int_0^1 \mu_1(\Theta J(sx)\Theta^{-1})ds\right)|\Theta x|_1 \tag{90}$$

$$\geq \tilde{c}|\Theta x|_1 \tag{91}$$

where the first inequality follows from the fact that $|Ax| \geq -\mu(A)|x|$ for all A and all x where $|\cdot|$ and $\mu(\cdot)$ are any vector norm and corresponding matrix measure, and the second inequality follows from the fact that $\mu(A+B) \leq \mu(A) + \mu(B)$ for all A, B (see [14] for a proof of both these facts). □

8 An algorithm for computing separable Lyapunov functions

In this section, we briefly discuss an efficient and scalable algorithm for computing $\theta(x)$ and $\omega(x)$ in Theorems 1 and 2 when each element of $f(x)$ is a polynomial or rational function of x. Thus, the proposed approach provides an efficient means for computing sum-separable and max-separable Lyapunov functions of monotone systems.

Our proposed algorithm relies on *sum-of-squares (SOS) programming* [34, 35]. A polynomial $s(x)$ is a *sum-of-squares polynomial* if $s(x) = \sum_{i=1}^{r}(g_i(x))^2$ for some polynomials $g_i(x)$ for $i = 1, \ldots, r$. A *SOS* *feasibility problem* consists in finding a collection of polynomials $p_i(x)$ for $i = 1, \dots, N$ and a collection of SOS polynomials $s_i(x)$ for $i = 1, \dots, M$ such that

$$a_{0,j} + \sum_{i=1}^{N} p_i(x) a_{i,j}(x) + \sum_{i=1}^{M} s_i(x) b_{i,j}(x) \tag{92}$$

$$\text{are SOS polynomials for } j = 1, \dots, J \tag{93}$$

for fixed polynomials $a_{0,j}(x)$, $a_{i,j}(x)$, and $b_{i,j}(x)$ for all i, j. The set of polynomials $\{p_i(x)\}_{i=1}^N$ and $\{s_i(x)\}_{i=1}^M$ satisfying (92)–(93) forms a convex set, and by fixing the degree of these polynomials, we arrive at a finite dimensional convex optimization problem. There exists efficient computational toolboxes that convert SOS feasibility programs into standard semi-definite programs (SDP) [37]. SOS programming has led to efficient computational algorithms for a number of controls-related applications such as searching for polynomial Lyapunov functions [33], underapproximating regions of attraction [50], and safety verification [10].

Here, we will present a SOS feasibility problem that is sufficient for finding $\theta(x)$ or $\omega(x)$ satisfying Theorem 1 or 2. To that end, recall that in the hypotheses of these theorems, we assume $\mathcal{X}$ is rectangular. In this section, for simplicity, we assume $\mathcal{X}$ is a closed set with nonzero measure so that $\mathcal{X} = \mathbf{cl}\{x : a_i < x_i < b_i \text{ for all } i\}$ for appropriately defined a_i and b_i where possibly $a_i = -\infty$ and/or $b_i = \infty$ and where $\mathbf{cl}$ denotes closure. Equivalently, $\mathcal{X} = \{x : d(x) \geq 0\}$ where $d(x) = \begin{bmatrix} d_1(x_1) & \dots & d_n(x_n) \end{bmatrix}^T$ and

$$d_i(x_i) = \begin{cases} x_i - a_i & \text{if } a_i \neq -\infty \text{ and } b_i = \infty \\ (x_i - a_i)(b_i - x_i) & \text{if } a_i \neq -\infty \text{ and } b_i \neq \infty \\ b_i - x_i & \text{if } a_i = -\infty \text{ and } b_i \neq \infty \\ 0 & \text{if } a_i = -\infty \text{ and } b_i = \infty. \end{cases} \tag{94}$$

We assume that the equilibrium x^* is known so that Proposition 2 applies. If $\sigma(x) = \begin{bmatrix} \sigma_1(x) & \dots & \sigma_n(x) \end{bmatrix}^T$ where each $\sigma_j(x)$, $j = 1 \dots n$ is a SOS polynomial, we call $\sigma(x)$ a *SOS n-vector.*

Proposition 5. *Let* (6) *be a monotone system with equilibrium x^* and rectangular domain $\mathcal{X} = \{x : d(x) \geq 0\}$ where $d(x) = \begin{bmatrix} d_1(x_1) & \dots & d_n(x_n) \end{bmatrix}^T$. Suppose each $f_i(x)$ is polynomial. Then the following is a SOS feasibility problem, and a feasible solution provides $\theta(x)$ satisfying the conditions of Theorem 1:*

For some fixed $\epsilon > 0$,
Find:
Polynomials $\theta_i(x_i)$, for $i = 1, \dots, n$
SOS polynomials $s_i(x_i)$ for $i = 1, \dots, n$
SOS n-vectors $\sigma^i(x)$ for $i = 1, \dots, n$
Such that:

$$(\theta_i(x_i) - \epsilon) - s_i(x_i) d_i(x_i) \quad \text{is a SOS polynomial for } i = 1, \dots, n \tag{95}$$

$$-(\theta(x)^T J(x) + \dot{\theta}(x)^T)_i - \sigma^i(x)^T d(x) \quad \text{is a SOS polynomial for } i = 1, \dots, n \tag{96}$$

$$-\theta(x^*)^T J(x^*) - \epsilon \geq 0 \tag{97}$$

where $(\theta(x)^T J(x) + \dot{\theta}(x)^T)_i$ denotes the i-th entry of $\theta(x)^T J(x) + \dot{\theta}(x)^T$.

Proof. Note that $(\theta(x)^T J(x) + \dot{\theta}(x)^T)_i$ is a polynomial in x for which the decision variables of the SOS feasibility problem appear linearly. In addition, (97) is a linear constraint on the coefficients of $\theta(x)$. Thus, (95)–(97) is a well-defined SOS feasibility problem.

Next, we claim that (95) is sufficient for $\theta(x) \geq \epsilon \mathbf{1}$ for all $x \in \mathcal{X}$. To prove the claim, suppose (95) is satisfied so that $(\theta_i(x_i) - \epsilon) - s_i(x_i) d_i(x_i)$ is a SOS polynomial and consider $x \in \mathcal{X}$ so that $d(x) \geq 0$. Since $s_i(x_i)$ is a SOS polynomial for all i, we have that also $s_i(x_i) d_i(x_i) \geq 0$ so that $\theta_i(x_i) \geq \epsilon + s_i(x_i) d_i(x_i) \geq \epsilon$.

A similar argument shows that if (96) holds for all i, then condition (1) of Theorem 1 holds, that is, (10) holds for all $x \in \mathcal{X}$. Indeed, suppose $-(\theta(x)^T J(x) + \dot{\theta}(x)^T)_i - \sigma^i(x)^T d(x)$ is a SOS polynomial and consider $x \in \mathcal{X}$ so that $d(x) \geq 0$. Since $\sigma^i(x)$ is a SOS n-vector, $\sigma^i(x)^T d(x) \geq 0$ and thus $-(\theta(x)^T J(x) + \dot{\theta}(x)^T)_i \geq 0$.

Finally, (97) implies (19), which is sufficient for condition (2) of Theorem 1 via Proposition 2. □

The technique employed in (95) and (96) to ensure that the conditions of Theorem 1 hold whenever $d(x) \geq 0$ is similar to the $\mathcal{S}$-procedure used to express constraints on quadratic forms as linear matrix inequalities [4, p. 23] and is common in applications of SOS programming to systems and control theory.

Remark 4. *A symmetric proposition holds establishing a SOS feasibility program sufficient for computing $\omega(x)$ that satisfies the conditions of Theorem 2. We omit an explicit form for this SOS program due to space constraints.*

Example 1 (Cont.)**.** *We consider the system from Example 1 and seek to compute a sum-separable Lyapunov function using the SOS program of Section 8. We let $\epsilon = 0.01$ and search for $\theta_1(x_1)$ and $\theta_2(x_2)$ that are polynomials of up to second-order. We consider SOS polynomials that are 0-th order, that is, all SOS polynomials are considered to be positive constants, which proves sufficient for this example. The SOS program requires 0.26 seconds of computation time and returns*

$$\theta_1(x) = 1.7429 \tag{98}$$

$$\theta_2(x) = x_2^2 + 1.3793x_2 + 1.9503 \tag{99}$$

where parameters have been scaled so that the leading coefficient of $\theta_2(x_2)$ is 1 since scaling $\theta(x)$ does not affect the validity of the resulting SOS program or the conditions of Theorem 1. Then, (12) *and* (13) *give the following sum-separable Lyapunov functions:*

$$V(x) = 1.7429x_1 + \frac{1}{3}x_2^3 + \frac{1.3793}{2}x_2^2 + 1.9503x_2 \tag{100}$$

$$V(x) = 1.7429\left|-x_1 + x_2^2\right| + x_2^3 + 1.3793x_2^2 + 1.9503. \tag{101}$$

■

9 Applications

In this section, we present several applications of our main results. First, we establish a technical result that will be useful for constructing Lyapunov functions as the limit of a sequence of contraction metrics.

Proposition 6. *Let $x^* \in \mathcal{X}$ be an equilibrium for* (6)*. Suppose there exists a sequence of global Lyapunov functions $V^i : \mathcal{X} \to \mathbb{R}_{\geq 0}$ for* (6) *that converges locally uniformly to $V(x) := \lim_{i\to\infty} V^i(x)$. If $V(x)$ is positive definite and globally proper (see Definition 2) then $V(x)$ is also a global Lyapunov function for* (6)*.*

Proof. Note that x^* is globally asymptotic stable since there exists a global Lyapunov function, and consider some $x_0 \in \mathcal{X}$. Asymptotic stability of x^* implies there exists a bounded set Ω for which $\phi(t, x_0) \in \Omega \subseteq \mathcal{X}$ for all $t \geq 0$. For $i = 1, \dots, n$, we have $V^i(\phi(t, x_0))$ is nonincreasing in t and $\lim_{t\to\infty} V^i(\phi(t, x_0)) = 0$. Local uniform convergence establishes $V(x)$ is continuous, $V(\phi(t, x_0))$ is nonincreasing in t, and $\lim_{t\to\infty} V(\phi(t, x_0)) = 0$, and thus condition (3) of Definition 2 holds. Under the additional hypotheses of the proposition, we have that $V(x)$ is therefore a global Lyapunov function. □

Note that a sequence $V^i(x)$ arising from a sequence of weighted contraction metrics, *i.e.*, $V^i(x) = |P_i(x - x^*)|$ or $V^i(x) = |P_i f(x)|$ for P_i converging to some nonsingular P, satisfies the conditions of Proposition 6.

The following example is inspired by [15, Example 3].

Example 4 (Comparison system)**.** *Consider the system*

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{102}$$

$$\dot{x}_2 = -2x_2 - x_2^2 + \gamma(x_1)^2 \tag{103}$$

evolving on $\mathcal{X} = \mathbb{R}^2_{\geq 0}$ *where* $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is strictly increasing and satisfies* $\gamma(0) = 0$, $\bar{\gamma} := \lim_{\sigma\to\infty} \gamma(\sigma) < 1$, *and* $\gamma'(\sigma) \leq \frac{1}{(1+\sigma)^2}$. *Consider the change of coordinates* $(\eta_1, \eta_2) = (\log(1 + x_1), x_2)$ *so that*

$$\dot{\eta}_1 = \frac{1}{1 + x_1}(-x_1 + x_1 x_2) \tag{104}$$

where we substitute $(x_1, x_2) = (e^{\eta_1} - 1, \eta_2)$. *Then*

$$\dot{\eta}_1 \leq -\beta(e^{\eta_1} - 1) + \eta_2 \tag{105}$$

where $\beta(\sigma) = \sigma/(1 + \sigma)$. *Introduce the comparison system*

$$\dot{\xi}_1 = -\beta(e^{\xi_1} - 1) + \xi_2 \tag{106}$$
$$\dot{\xi}_2 = -2\xi_2 - \xi_2^2 + \gamma(e^{\xi_1} - 1)^2 \tag{107}$$

evolving on $\mathbb{R}^2_{\geq 0}$. *The comparison principle (see, e.g., [15]) ensures that asymptotic stability of the origin for the comparison system* (106)–(107) *implies asymptotic stability of the origin of the* (η_1, η_2) *system, which in turn establishes asymptotic stability of the origin for* (102)–(103). *The Jacobian of* (106)–(107) *is given by*

$$J(\xi) = \begin{pmatrix} -e^{\xi_1}\beta'(e^{\xi_1} - 1) & 1 \\ 2e^{\xi_1}\gamma(e^{\xi_1} - 1)\gamma'(e^{\xi_1} - 1) & -2 - 2\xi_2 \end{pmatrix} \tag{108}$$

where $\beta'(\sigma) = \frac{1}{(1+\sigma)^2}$. *Let* $v = (2\bar{\gamma} + \epsilon, 1)$ *where* ϵ *is chosen small enough so that* $c_1 := (2\bar{\gamma} + \epsilon - 2) < 0$. *It follows that*

$$v^T J(\xi) \leq (-\epsilon e^{-\xi_1}, c_1) < 0 \quad \forall \xi. \tag{109}$$

Applying Corollary 1 and noting Remark 2 , the origin of (102)–(103) *and* (106)–(107) *is globally asymptotically stable. Furthermore, we have the following state and flow sum-separable Lyapunov functions for the comparison system* (106)–(107)*:*

$$V(\xi) = (2\bar{\gamma} + \epsilon)\xi_1 + \xi_2 \tag{110}$$
$$V(\xi) = (2\bar{\gamma} + \epsilon)|\dot{\xi}_1| + |\dot{\xi}_2|. \tag{111}$$

Above, we understand $\dot{\xi}_1$ *and* $\dot{\xi}_2$ *to be shorthand for the equalities expressed in* (106)–(107).

Example 5 (Multiagent system)**.** *Consider the following system evolving on* $\mathcal{X} = \mathbb{R}^3$*:*

$$\dot{x}_1 = -\alpha_1(x_1) + \rho_1(x_3 - x_1) \tag{112}$$
$$\dot{x}_2 = \rho_2(x_1 - x_2) + \rho_3(x_3 - x_2) \tag{113}$$
$$\dot{x}_3 = \rho_4(x_2 - x_3) \tag{114}$$

where we assume $\alpha_1 : \mathbb{R} \to \mathbb{R}$ *is strictly increasing and satisfies* $\alpha(0) = 0$ *and* $\alpha_1'(\sigma) \geq \underline{c}_0$ *for some* $\underline{c}_0 > 0$ *for all* σ, *and each* $\rho_i : \mathbb{R} \to \mathbb{R}$ *is strictly increasing and satisfies* $\rho_i(0) = 0$. *Furthermore, for* $i = 1, 3$, $\rho_i'(\sigma) \leq \overline{c}_i$ *for some* $\overline{c}_i > 0$ *for all* σ, *and for* $i = 2, 4$, $\rho_i'(\sigma) \geq \underline{c}_i$ *for some* $\overline{c}_i > 0$ *for all* σ.

For example, x_1, x_2, *and* x_3 *may be the position of three vehicles, for which the dynamics* (112)–(114) *are a rendezvous protocol whereby agent 1 moves towards agent 3 at a rate dependent on the distance* $x_3 - x_1$ *as determined by* ρ_1, etc. *Additionally, agent 1 navigates towards the origin according to* $-\alpha_1(x_1)$. *Computing the Jacobian, we obtain*

$$J(x) = \begin{pmatrix} -\alpha'(x_1) - \rho_1'(z_{31}) & 0 & \rho_1'(z_{31}) \\ \rho_2'(z_{12}) & -\rho_2'(z_{12}) - \rho_3'(z_{32}) & \rho_3'(z_{32}) \\ 0 & \rho_4'(z_{23}) & -\rho_4'(z_{23}) \end{pmatrix} \tag{115}$$

where $z_{ij} := x_i - x_j$. *Let* $w = (1, 1+\epsilon_1, 1+\epsilon_1+\epsilon_2)^T$ *where* $\epsilon_1 > 0$ *and* $\epsilon_2 > 0$ *are chosen to satisfy*

$$\underline{c}_0 > (\epsilon_1+\epsilon_2)\overline{c}_1 \quad \textit{and} \quad \epsilon_1\underline{c}_2 > \epsilon_2\overline{c}_3. \tag{116}$$

We then have $J(x)w \leq c\mathbf{1}$ *for all* x *for* $c = \max\{(\epsilon_1+\epsilon_2)\overline{c}_1 - \underline{c}_0, \epsilon_2\overline{c}_3 - \epsilon_1\underline{c}_2, -\epsilon_2\underline{c}_4\} < 0$. *Thus, the origin of* (112)–(114) *is globally asymptotically stable by Corollary 2. Furthermore,*

$$V(x) = \max\{|x_1|, (1+\epsilon_1)^{-1}|x_2|, (1+\epsilon_1+\epsilon_2)^{-1}|x_3|\}, \tag{117}$$

$$V(x) = \max\{|\dot{x}_1|, (1+\epsilon_1)^{-1}|\dot{x}_2|, (1+\epsilon_1+\epsilon_2)^{-1}|\dot{x}_3|\} \tag{118}$$

are state and flow max-seperable Lyapunov functions where we interpret $\dot{x}_i$ *as shorthand for the equalities expressed in* (112)–(114). *Since we may take* ϵ_1 *and* ϵ_2 *arbitrarily small satisfying* (116), *using Proposition 6 we have also the following choices for Lyapunov functions:*

$$V(x) = \max\{|x_1|, |x_2|, |x_3|\}, \tag{119}$$

$$V(x) = \max\{|\dot{x}_1|, |\dot{x}_2|, |\dot{x}_3|\}. \tag{120}$$

The flow max-separable Lyapunov functions (118) *and* (120) *are particularly useful for multiagent vehicular networks where it often easier to measure each agent's velocity rather than absolute position.*

In Example 5, choosing $w = \mathbf{1}$, we have $J(x)w \leq 0$ for all x, however this is not enough to establish asymptotic stability using Corollary 2. Informally, choosing w as in the example distributes the extra negativity of $-\alpha'(x_1)$ among the columns of $J(x)$. Nonetheless, Proposition 6 implies that choosing $w = \mathbf{1}$ indeed leads to a valid Lyapunov function.

The above example generalizes to systems with many agents interacting via arbitrary directed graphs, as does the principle of distributing extra negativity along diagonal entries of the Jacobian as discussed in Section 10.

Example 6 (Traffic flow)**.** *A model of traffic flow along a freeway with no onramps is obtained by spatially partitioning the freeway into* n *segments such that traffic flows from segment* i *to* $i+1$, $x_i \in [0, \bar{x}_i]$ *is the density of vehicles occupying link* i, *and* $\bar{x}_i$ *is the capacity of link* i. *A fraction* $\beta_i \in (0,1]$ *of the flow out of link* i *enters link* $i+1$. *The remaining* $1-\beta_i$ *fraction is assumed to exit the network via,* e.g., *unmodeled offramps. Associated with each link is a continuously differentiable* demand *function* $D_i : [0, \bar{x}_i] \to \mathbb{R}_{\geq 0}$ *that is strictly increasing and satisfies* $D_i(0) = 0$, *and a continuously differentiable* supply *function* $S_i : [0, \bar{x}_i] \to \mathbb{R}_{\geq 0}$ *that is strictly decreasing and satisfies* $S_i(\bar{x}_i) = 0$. *Flow from segment to segment is restricted by upstream demand and downstream supply, and the change in density of a link is governed by mass conservation:*

$$\dot{x}_1 = \min\{\delta_1, S_1(x_1)\} - \frac{1}{\beta_1} g_1(x_1, x_2) \tag{121}$$

$$\dot{x}_i = g_{i-1}(x_{i-1}, x_i) - \frac{1}{\beta_i} g_i(x_i, x_{i+1}), \quad i = 2, \ldots, n-1 \tag{122}$$

$$\dot{x}_n = g_{n-1}(x_{n-1}, x_n) - D_n(x_n) \tag{123}$$

for some $\delta_1 > 0$ *where, for* $i = 1, \ldots, n-1$,

$$g_i(x_i, x_{i+1}) = \min\{\beta_i D_i(x_i), S_{i+1}(x_{i+1})\}. \tag{124}$$

Let $\delta_i \triangleq \delta_1 \prod_{j=1}^{i-1} \beta_j$ *for* $i = 2, \ldots, n$. *If* $d_i^{-1}(\delta_i) < s_i^{-1}(\delta_i)$ *for all* i, *then* δ_1 *is said to be* feasible *and* $x_i^* := d_i^{-1}(\delta_i)$ *constitutes the unique equilibrium.*

Let ∂_i *denote differentiation with respect to the* i*th component of* x, *that is,* $\partial_i g(x) := \frac{\partial g}{\partial x_i}(x)$ *for a function* $g(x)$. *The dynamics* (121)–(123) *define a system* $\dot{x} = f(x)$ *for which* f *is continuous but only piecewise differentiable. Nonetheless, the results developed above apply for this case, and, in the sequel, we interpret statements involving derivatives to hold wherever the derivative exists.*

Notice that $\partial_i g_i(x_i, x_{i+1}) \geq 0$ and $\partial_{i+1} g_i(x_i, x_{i+1}) \leq 0$. Define $g_0(x_1) := \min\{\delta_1, S_1(x_1)\}$. The Jacobian, where it exists, is given by

$$J(x) = \begin{pmatrix} \partial_1 g_0 - \frac{1}{\beta_1}\partial_1 g_1 & -\frac{1}{\beta_1}\partial_2 g_1 & 0 & 0 & \cdots & 0 \\ \partial_1 g_1 & \partial_2 g_1 - \frac{1}{\beta_2}\partial_2 g_2 & -\frac{1}{\beta_2}\partial_3 g_2 & 0 & \cdots & 0 \\ 0 & \partial_2 g_2 & \partial_3 g_2 - \frac{1}{\beta_3}\partial_3 g_3 & -\frac{1}{\beta_3}\partial_4 g_3 & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \partial_{n-1} g_{n-1} & \partial_n g_{n-1} - \partial_n D_n(x_n) \end{pmatrix}, \tag{125}$$

which is seen to be Metzler. Let

$$\tilde{v} = \left(1, \beta_1^{-1}, (\beta_1\beta_2)^{-1}, \ldots, (\beta_1\beta_2\cdots\beta_{n-1})^{-1}\right)^T. \tag{126}$$

Then $\tilde{v}^T J(x) \leq 0$ for all x. Moreover, there exists $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}, 0)$ with $\epsilon_i > \epsilon_{i+1}$ for each i such that $v := \tilde{v} + \epsilon$ satisfies

$$v^T J(x) \leq 0 \quad \forall x \tag{127}$$

$$v^T J(x^*) < 0. \tag{128}$$

Such a vector ϵ is constructed using a technique similar to that used in Example 5. In particular, the sum of the nth column of diag($\tilde{v}$)$J(x)$ is strictly negative because $-\partial_n D_n(x_n) < 0$, and this excess negativity is used to construct v such that (127)–(128) holds. A particular choice of ϵ such that (127)–(128) holds depends on bounds on the derivative of the demand functions D_i, but it is possible to choose ϵ arbitrarily small. Corollary 1 establishes asymptotic stability, and Proposition 6 gives the following sum-separable Lyapunov functions:

$$V(x) = \sum_{i=1}^{n} \left(x_i \prod_{j=1}^{i-1} \beta_j \right), \tag{129}$$

$$V(x) = \sum_{i=1}^{n} \left(|\dot{x}_i| \prod_{j=1}^{i-1} \beta_j \right), \tag{130}$$

where we interpret $\dot{x}_i$ according to (121)–(123).

In traffic networks, it is often easier to measure traffic flow rather than traffic density. Thus, (130) is a practical Lyapunov function indicating that the (weighted) total absolute net flow throughout the network decreases over time.

In [9], a result similar to that of Example 6 is derived for possibly infeasible input flow and traffic flow network topologies where merging junctions with multiple incoming links are allowed. The proof considers a flow sum-separable Lyapunov function similar to (130) and appeals to LaSalle's invariance principle rather than Proposition 6.

10 Discussion

10.1 Relationship to ISS small-gain conditions

In this section, we briefly discuss the relationship between the main results of this paper and small-gain conditions for interconnected input-to-state stable (ISS) systems. While there appears to be a number of interesting connections between the results presented here and the extensive literature on networks of ISS systems, see, *e.g.*, [20, 21] for some recent results, a complete characterization of this relationship is outside the scope of this paper and will be the subject of future research. Nonetheless, we highlight how Theorems 1 and 2 provide a Jacobian-based perspective to ISS system analysis.

Consider N interconnected systems with dynamics $\dot{x}_i = f_i(x_1, \ldots, x_N)$ for $x_i \in \mathbb{R}^{n_i}$ and suppose each system satisfies an input-to-state stability (ISS) condition [43] whereby there exists ISS Lyapunov functions V_i [48] satisfying

$$\frac{\partial V_i}{\partial x_i}(x_i)f_i(x) \leq -\alpha_i(V_i(x_i)) + \sum_{i \neq j} \gamma_{ij}(V_j(x_j)) \tag{131}$$

where each α_i and γ_{ij} is a class $\mathcal{K}_\infty$ function[1]. We obtain a monotone comparison system

$$\dot{\xi} = g(\xi), \qquad g_i(\xi) = -\alpha_i(\xi_i) + \sum_{j \neq i} \gamma_{ij}(\xi_j) \tag{132}$$

evolving on $\mathbb{R}^n_{\geq 0}$ for which asymptotic stability of the origin implies asymptotic stability of the original system [41].

It is shown in [12, Section 4.3] that if $\gamma_{ij}(s) = k_{ij}h_j(s)$ and $\alpha_i(s) = a_i h_i(s)$ for some collection of constants $c_{ij} \geq 0$, $a_j > 0$ and $\mathcal{K}_\infty$ functions h_i for all $i, j = 1, \ldots, n$, and there exists a vector v such that $v^T(-A+C) < 0$ where $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $[C]_{ij} = c_{ij}$, then $V(x) = v^T \begin{bmatrix} V_1(x_1) & \ldots V_n(x_n) \end{bmatrix}^T$ is an ISS Lyapunov function for the composite system. Indeed, in this case, and considering the comparison system (132), we see that

$$\frac{\partial g}{\partial \xi}(\xi) = (-A + C)\mathrm{diag}(h_1'(\xi_1), \cdots, h_n'(\xi_n)) \tag{133}$$

where $h_i'(\xi_i) \geq 0$ so that, if $v^T(-A+C) < 0$, then also $v^T \frac{\partial g}{\partial \xi}(\xi) \leq 0$ for all ξ. If also $h_i'(0) > 0$ for all i, then $v^T \frac{\partial g}{\partial \xi}(0) < 0$ so that Corollary 1 and Remark 2 imply the sum-separable Lyapunov function $v^T\xi$, providing a contraction theoretic interpretation of this result. The case of $N = 2$ was first investigated in [23] where it is assumed without loss of generality that $a_1 = a_2 = 1$ and it is shown that if $c_{12}c_{21} < 1$, then $v_1V_1(x_1) + v_2V(x_2)$ is a Lyapunov function for the original system for any $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T > 0$ satisfying $v_1c_{12} < v_2$ and $v_2c_{21} < v_1$. These conditions are equivalent to $v^T(-I + C) < 0$.

Alternatively, in [13, 41], it is shown that if there exists a function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ with each component ρ_i belonging to class $\mathcal{K}_\infty$ such that $g(\rho(r)) < 0$ for all $r > 0$, then the origin is asymptotically stable and $V(\xi) := \max_i\{\rho_i^{-1}(\xi_i)\}$ is a Lyapunov function. If the conditions of Corollary 2 hold for the comparison system for some w, we may choose $\rho(r) = rw$. Indeed, we have

$$g(rw) = \int_0^1 \frac{\partial g}{\partial \xi}(\sigma rw)rw \; d\sigma < 0 \quad \forall r > 0. \tag{134}$$

For this case, $V(\xi) = \max_i\{\rho_i^{-1}(\xi_i)\} = \max_i\{\xi_i/w_i\}$, recovering (34).

10.2 Generalized contraction and compartmental systems

We now discuss the relationship between the results presented here and additional results for contractive systems in the literature. First, we comment on the relationship between Corollary 4 of Theorem 3 as well as Proposition 6 and a generalization of contraction theory recently developed in [30, 47] where exponential contraction between any two trajectories is required only after an arbitrarily small amount of time, an arbitrarily small overshoot, or both. In [30, Corollary 1], it is shown that if a system is contractive with respect to a sequence of norms convergent to some norm, then the system is generalized contracting with respect to that norm, a result analogous to Proposition 6. In [30], conditions on the sign structure of the Jacobian are obtained that ensure the existence of such a sequence of weighted ℓ_1 or ℓ_∞ norms. These conditions are a generalization of the technique in Example 5 and Example 6 in Section 9 where small ϵ is used to distribute excess negativity.

Furthermore, it is shown in [29, 31] that a ribosome flow model for gene translation is monotone and nonexpansive with respect to a weighted ℓ_1 norm, and additionally is contracting on a subset of its domain.

[1] A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$, and $\lim_{r \to \infty} \alpha(r) = \infty$.

Entrainment of solutions is proved by first showing that all trajectories reach the region of exponential contraction. Theorem 3 provides a different approach for studying entrainment by observing that the distance to the periodic trajectory strictly decreases in each period due to a neighborhood of contraction along the periodic trajectory.

Finally, we note that Metzler matrices with nonpositive column sums have also been called *compartmental* [22]. It has been shown that if the Jacobian matrix is compartmental for all x, then $V(x) = |f(x)|$ is a nonincreasing function along trajectories of (6) [22, 27]; Proposition 3 recovers this observation when considering (42) with $c = 0$, $\Theta(x) \equiv I$, and $|\cdot|$ taken to be the ℓ_1 norm.

11 Conclusions

We have investigated monotone systems that are also contracting with respect to a weighted ℓ_1 norm or ℓ_∞ norm. In the case of the ℓ_1 (respectively, ℓ_∞) norm, we provided a condition on the weighted column (respectively, row) sums of the Jacobian matrix for ensuring contraction. When the norm is state-dependent, these conditions include an additive term that is the time derivative of the weights. This construction leads to a pair of sum-separable (respectively, max-separable) Lyapunov functions. The first Lyapunov function pair is separable along the state of the system while the second is *agent-separable*, that is, each constituent function depends on $f_i(x)$ in addition to x_i. When the weighted contractive norm is independent of state, the components of this Lyapunov function only depend on $f_i(x)$ and we say it is *flow-separable*. Such flow separable Lyapunov functions are especially relevant in applications where it is easier to measure the derivative of the system's state rather than measure the state directly.

In addition, we provided a computational algorithm to search for separable Lyapunov functions using our main results and sum-of-squares programming, and we demonstrated our results through several examples. We further highlighted some connections to stability results for interconnected input-to-state stable systems. These connections appear to be a promising direction for future work.

References

[1] Angeli, D. and Sontag, E. (2003). Monotone control systems. *IEEE Transactions on Automatic Control*, 48(10):1684–1698.

[2] Angeli, D. and Sontag, E. (2004). Interconnections of monotone systems with steady-state characteristics. In *Optimal control, stabilization and nonsmooth analysis*, pages 135–154. Springer.

[3] Bao, D., Chern, S.-S., and Shen, Z. (2012). *An introduction to Riemann-Finsler geometry*, volume 200. Springer Science & Business Media.

[4] Boyd, S., Ghaoui, L. E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM.

[5] Clarke, F. H. (1990). *Optimization and nonsmooth analysis*, volume 5. Siam.

[6] Como, G., Lovisari, E., and Savla, K. (2015). Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. *IEEE Transactions on Control of Network Systems*, 2(1):57–67.

[7] Coogan, S. (2016). Separability of Lyapunov functions for contractive monotone systems. In *IEEE Conference on Decision and Control*, pages 2184–2189.

[8] Coogan, S. and Arcak, M. (2014). Dynamical properties of a compartmental model for traffic networks. In *Proceedings of the 2014 American Control Conference*, pages 2511–2516.

[9] Coogan, S. and Arcak, M. (2015a). A compartmental model for traffic networks and its dynamical behavior. *IEEE Transactions on Automatic Control*, 60(10):2698–2703.

[10] Coogan, S. and Arcak, M. (2015b). A dissipativity approach to safety verification for interconnected systems. *IEEE Transactions on Automatic Control*, 60(6):1722–1727.

[11] Coogan, S. and Arcak, M. (2015c). Efficient finite abstraction of mixed monotone systems. In *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, pages 58–67.

[12] Dashkovskiy, S., Ito, H., and Wirth, F. (2011). On a small gain theorem for ISS networks in dissipative lyapunov form. *European Journal of Control*, 17(4):357–365.

[13] Dashkovskiy, S. N., Rüffer, B. S., and Wirth, F. R. (2010). Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118.

[14] Desoer, C. and Vidyasagar, M. (2008). *Feedback systems: Input-output properties*. Society for Industrial and Applied Mathematics.

[15] Dirr, G., Ito, H., Rantzer, A., and Rüffer, B. (2015). Separable Lyapunov functions for monotone systems: constructions and limitations. *Discrete Contin. Dyn. Syst. Ser. B*.

[16] Forni, F. and Sepulchre, R. (2014). A differential Lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3):614–628.

[17] Gomes, G., Horowitz, R., Kurzhanskiy, A. A., Varaiya, P., and Kwon, J. (2008). Behavior of the cell transmission model and effectiveness of ramp metering. *Transportation Research Part C: Emerging Technologies*, 16(4):485–513.

[18] Hirsch, M. W. (1983). Differential equations and convergence almost everywhere in strongly monotone semiflows. *Contemporary Mathematics*, 17:267–285.

[19] Hirsch, M. W. (1985). Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere. *SIAM Journal on Mathematical Analysis*, 16(3):423–439.

[20] Ito, H., Dashkovskiy, S., and Wirth, F. (2012). Capability and limitation of max-and sum-type construction of Lyapunov functions for networks of iISS systems. *Automatica*, 48(6):1197–1204.

[21] Ito, H., Jiang, Z. P., Dashkovskiy, S. N., and Rüffer, B. S. (2013). Robust stability of networks of iISS systems: Construction of sum-type lyapunov functions. *IEEE Transactions on Automatic Control*, 58(5):1192–1207.

[22] Jacquez, J. A. and Simon, C. P. (1993). Qualitative theory of compartmental systems. *SIAM Review*, 35(1):43–79.

[23] Jiang, Z.-P., Mareels, I. M., and Wang, Y. (1996). A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211 – 1215.

[24] Lohmiller, W. and Slotine, J.-J. E. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696.

[25] Lovisari, E., Como, G., Rantzer, A., and Savla, K. (2014a). Stability analysis and control synthesis for dynamical transportation networks. *arXiv:1410.5956*.

[26] Lovisari, E., Como, G., and Savla, K. (2014b). Stability of monotone dynamical flow networks. In *Proceedings of the 53rd Conference on Decision and Control*, pages 2384–2389.

[27] Maeda, H., Kodama, S., and Ohta, Y. (1978). Asymptotic behavior of nonlinear compartmental systems: nonoscillation and stability. *IEEE Transactions on Circuits and Systems*, 25(6):372–378.

[28] Manchester, I. R. and Slotine, J.-J. E. (2017). On existence of separable contraction metrics for monotone nonlinear systems. *arxiv:1704.02676*.

[29] Margaliot, M., Sontag, E. D., and Tuller, T. (2014). Entrainment to periodic initiation and transition rates in a computational model for gene translation. *PloS one*, 9(5):e96039.

[30] Margaliot, M., Sontag, E. D., and Tuller, T. (2016). Contraction after small transients. *Automatica*, 67:178–184.

[31] Margaliot, M. and Tuller, T. (2012). Stability analysis of the ribosome flow model. *IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB)*, 9(5):1545–1552.

[32] M.Vidyasagar (2002). *Nonlinear System Analysis*. Society for Industrial and Applied Mathematics.

[33] Papachristodoulou, A. and Prajna, S. (2002). On the construction of Lyapunov functions using the sum of squares decomposition. In *Proceedings of the 41st IEEE Conference on Decision and Control*, volume 3, pages 3482–3487 vol.3.

[34] Parrilo, P. (2000). *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology.

[35] Parrilo, P. A. (2003). Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming Ser. B*, 96(2):293–320.

[36] Pavlov, A., Pogromsky, A., van de Wouw, N., and Nijmeijer, H. (2004). Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems & Control Letters*, 52(3):257–261.

[37] Prajna, S., Papachristodoulou, A., Seiler, P., and Parrilo, P. A. (2004). *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*.

[38] Rantzer, A. (2015). Scalable control of positive systems. *European Journal of Control*, 24:72–80.

[39] Rantzer, A., Ruffer, B., and Dirr, G. (2013). Separable Lyapunov functions for monotone systems. In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pages 4590–4594.

[40] Raveh, A., Margaliot, M., Sontag, E. D., and Tuller, T. (2016). A model for competition for ribosomes in the cell. *Journal of The Royal Society Interface*, 13(116):20151062.

[41] Rüffer, B. S., Kellett, C. M., and Weller, S. R. (2010). Connection between cooperative positive systems and integral input-to-state stability of large-scale systems. *Automatica*, 46(6):1019–1027.

[42] Smith, H. L. (1995). *Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems*. American Mathematical Society.

[43] Sontag, E. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443.

[44] Sontag, E. D. (1998). *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, second edition.

[45] Sontag, E. D. (2007). Monotone and near-monotone biochemical networks. *Systems and Synthetic Biology*, 1(2):59–87.

[46] Sontag, E. D. (2010). Contractive systems with inputs. In *Perspectives in Mathematical System Theory, Control, and Signal Processing*, pages 217–228. Springer.

[47] Sontag, E. D., Margaliot, M., and Tuller, T. (2014). On three generalizations of contraction. In *IEEE 53rd Annual Conference on Decision and Control (CDC)*, pages 1539–1544.

[48] Sontag, E. D. and Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359.

[49] Sootla, A. (2016). Construction of max-separable Lyapunov functions for monotone systems using the Koopman operator. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 6512–6517.

[50] Topcu, U., Packard, A., Seiler, P., and Wheeler, T. (2007). Stability region analysis using simulations and sum-of-squares programming. In *Proceedings of the American Control Conference*, pages 6009–6014.