# Discontinuous Barrier Functions for Piecewise Continuous Dynamics

Carmen Jimenez Cortes[1], Makhin Thitsa[2], and Samuel Coogan[3]

*Abstract*— **We propose a novel characterization of piecewise-defined barrier functions for certifying forward invariant sets of piecewise continuous dynamical systems. Forward invariance is established by checking two conditions: the first condition is a usual barrier-type inequality on the interior of each piece, and the second condition imposes an appropriate interaction of the tangent cone and vector field at the boundary between pieces. We then show that this separation is especially well suited for constructing discontinuous barrier functions that are an appropriate generalization of high-order control barrier functions to the piecewise setting and can be used to construct controllers for forward invariance. In particular, the tangent cone condition at the boundary of pieces does not depend on the particular control strategy and can be checked, e.g., offline, while standard online methods can be used to enforce the barrier-type inequality on the interior of pieces.**

## I. Introduction

Safety of control systems modeled as a constraint on the system's state arises in applications ranging from collision avoidance [1] to motion planning [2] where safety is enforced with a barrier at the constraint boundary. Such approaches use barrier functions or control barrier functions (CBFs) [3]–[6] to enforce forward invariance of the constraint set. Most of the existing literature focuses on smooth functions as candidate barrier functions or CBFs and controllers that are at least Lipschitz continuous. However, nonsmooth candidate functions arise naturally in many applications and often give rise to discontinuous control strategies.

Discontinuous control strategies are used in [7], where the authors proved forward invariance of the composed set obtained as the intersection of multiple sets. Boolean compositions of level sets of continuously differentiable functions are considered in [8], [9] and applied to multiagent robotic systems in [10]. This approach requires the barrier function to still be continuous and uses a set-valued Lie derivative construction to obtain a sufficient condition for forward invariance that can be conservative in practice. The paper [11] provides general sufficient conditions for forward invariance of hybrid inclusions and allows for vector-valued barrier functions that are only locally Lipschitz using the Clarke generalized gradient. In [12], sufficient conditions for forward invariance of a set defined by a possibly discontinuous barrier function are given using the proximal subdifferential of the barrier. These conditions are general and applicable to hybrid inclusions but require computing the proximal subdifferential at all points in a neighborhood of the boundary of the candidate invariant set, which might be challenging to check in practice. Moreover, [12] does not particularly address the case of high-order barrier functions, discussed next.

Given a candidate CBF, the system dynamics might be such that the control input does not affect, up to first order, the rate of change of the barrier, that is, the input does not appear in the first time derivative of the barrier function. In this case, the system is said to have high relative degree with respect to the barrier and standard CBF approaches for forward invariance are generally not applicable. A common remedy is to obtain an alternative CBF constructed from the original candidate CBF and CBF-type inequalities. The resulting new CBF is called a high-order CBF (HO-CBF) to emphasize its manner of construction [13]–[15], however, existing approaches exclusively assume sufficient smoothness of the barrier function.

In this paper we prove forward invariance of sets defined by the level sets of piecewise-defined barrier functions for piecewise continuous dynamical systems. These barrier functions arise when we extend HO-CBFs theory to the nonsmooth setting, which leads to discontinuities. To do so, there are two conditions that need to be met: the first condition is barrier-type inequality on the interior of each piece of an open finite partition of the state space; meanwhile, the second condition imposes a well-behaved interaction between the tangent cone and the vector field at the boundary between each pair of pieces. To the best of our knowledge, this problem has not yet been addressed in the existing literature. Additionally, the second condition does not depend on the control input and can be checked offline, while the first one can be used to synthesize CBF-type controllers.

The rest of this paper is structured as follows: Section II includes key definitions, a review of smooth (control) barrier functions, and a review of high-order smooth control barrier functions. Section III contains the main results of this work, the definition and conditions for valid piecewise-defined (control) barrier functions, as well as a detailed example. Section IV extends the theory from Section III to discontinuous high-order CBFs and includes an additional example using the nonlinear dynamics of the Second-Order Unicycle. Finally, Section V summarizes the main contri-

This research was supported in part by the Office of Naval Research under award N00014-21-1-2759, the Ford Motor Company under the Ford-GT Alliance Program, the National Science Foundation under award #1749357, and The NASA University Leadership Initiative (grant #80NSSC20M0163). This article solely reflects the opinions and conclusions of its authors.

[1]Carmen Jimenez Cortes Ph.D. student in Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA ccortes9@gatech.edu

[2]Makhin Thitsa is with the school of Electrical and Computer Engineering, Mercer University, Macon, GA 31207, USA thitsa_m@mercer.edu

[3]Samuel Coogan is with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA sam.coogan@gatech.edu

butions of the paper and how they extend the scope of the current barrier functions and CBFs theory.

## II. Background Materials

### A. Tangent Cones, Filippov Solutions, and Invariance

Given a closed set $\mathcal{C} \subseteq \mathbb{R}^n$, the *Bouligand tangent cone* [16, Def. 2.2] to $\mathcal{C}$ at $x$ is defined as

$$\mathcal{T}_{\mathcal{C}}(x) = \left\{ z \mid \liminf_{\tau \to 0} \frac{\text{dist}(x + \tau z, \mathcal{C})}{\tau} = 0 \right\} \tag{1}$$

where $\text{dist}(\cdot, \cdot)$ is the distance function given by $\text{dist}(x, \mathcal{C}) = \min_{y \in \mathcal{C}} \|x - y\|$. Given a system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$, $x(t)$ is a *Filippov solution* [17] on $[0, T]$ if $x(t)$ is absolutely continuous on $[0, T]$ and for almost all $t \in [0, T]$ it holds that $\dot{x} \in K[f](x)$ where

$$K[f](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{co} f(B(x, \delta) \backslash N), \tag{2}$$

and $\bigcap_{\mu N = 0}$ denotes the intersection over all sets $N$ of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(x, \delta)$ is the ball of radius $\delta$ centered at $x$.

An *open finite partition* of $\mathbb{R}^n$ is a collection of disjoint, open, and connected sets $\{X_i\}_{i \in \mathcal{I}}$ for finite index set $\mathcal{I}$ whose closures cover $\mathbb{R}^n$, that is $\cup_{i \in \mathcal{I}} \text{cl}(X_i) = \mathbb{R}^n$ where $\text{cl}(\cdot)$ denotes closure. We denote the boundary of a part $X_i$ of the partition by $\partial X_i = \text{cl}(X_i) \backslash X_i$.

The system $\dot{x} = f(x)$ is *piecewise continuous* if there exists an open finite partition $\{X_i\}_{i \in \mathcal{I}}$ of $\mathbb{R}^n$ such that $f$ is continuous on $X_i$ for all $i \in \mathcal{I}$ and admits a continuous extension to $\text{cl}(X_i)$, denoted $f_{X_i}$. For piecewise continuous systems, there always exists a Filippov solution from each initial condition $x_0 \in \mathbb{R}^n$. Moreover, the set-valued map $K[f]$ is characterized as [18] $K[f](x) = co\{f_{X_i}(x) \mid x \in \text{cl}(X_i)\}$, a convex polyhedron in $\mathbb{R}^n$. In particular, $K[f](x) = \{f(x)\}$ if $f$ is continuous at $x$. Otherwise, if $f$ is discontinuous at $x$, then $x$ must be on the boundary of some parts of the partition, and $K[f]$ is the convex hull of the continuous extension of $f$ on all these parts. We will generally be interested in piecewise continuous systems that also admit unique solutions, which is often (but not exclusively) the case in practice, and restrict to such systems by assumption. Sufficient conditions ensuring uniqueness are provided in [18, Prop. 5].

A closed set $S$ is *forward invariant* for the system $\dot{x} = f(x)$ if for all $T > 0$, all $x_0 \in S$, and all Filippov solutions $x(t)$ on $[0, T]$ satisfying $x(0) = x_0$, *i.e.*, $x_0$ is a initial condition of the system, it holds that $x(t) \in S$ for all $t \in [0, T]$. This definition does not exclude the possibility of finite escape from within $S$, which can be ruled out with further assumptions such as a linear growth condition on $K[f]$ [16, Ch. 4, Cond. 1.2(c)] or compactness of $S$. Usually, this notion of forward invariance is called *strong* forward invariance since it requires all Filippov solutions to remain in $S$, and *weak* forward invariance is then used if some solution remains in $S$ for each initial condition from $S$. Since we restrict to systems with unique Filippov solutions, strong and weak notions of forward invariance become equivalent.

The following fundamental result, which is an adaptation of [16, Ch. 4, Thm. 2.10], connects all the concepts introduced above.

**Proposition 1.** *Let $\dot{x} = f(x)$ for $x \in \mathbb{R}^n$ be piecewise continuous and suppose it admits unique Filippov solutions. The closed set $S \subset \mathbb{R}^n$ is forward invariant if and only if*

$$K[f](x) \cap T_S(x) \neq \emptyset \quad \textit{for all } x \in S. \tag{3}$$

### B. Barrier Functions and Control Barrier Functions

Suppose now $S$ is defined as $S = \{x \mid h(x) \geq 0\}$ for some continuously differentiable function $h(x)$, called a *Barrier Function*, with the property that $h(x) = 0$ implies $\nabla h(x) \neq 0$. The boundary of $S$, denoted $\partial S = S \backslash \text{int}(S)$, is given by $\partial S = \{x \mid h(x) = 0\}$. Its tangent cone is given by $T_S(x) = \mathbb{R}^n$ for $x \in \text{int}(S)$, and $T_S(x) = \{v \mid \nabla h(x)^T v \geq 0\}$ for $x \in \partial S$. If, further, $f$ is Lipschitz continuous, it holds that

$$S \text{ is forward invariant } \iff \dot{h}(x) = \nabla h(x)^T f(x) \geq 0 \quad \text{for all } x \in \partial S, \tag{4}$$

which is classically known as Nagumo's Theorem. In the barrier function literature, the righthand condition is often relaxed to

$$\dot{h}(x) \geq -\alpha(h(x)) \quad \text{for all } x \in \mathbb{R}^n \tag{5}$$

for some locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$. This condition must hold for all $x$ rather than only on the boundary of $S$, which more readily leads to control design techniques. For example, consider the controlled system $\dot{x} = f(x) + g(x)u$, now with input $u \in \mathbb{R}^m$, and the goal of designing a feedback controller $\sigma(x)$ such that $S$ is forward invariant. Then, condition (5) leads to the design criterion that any Lipschitz continuous feedback controller $\sigma(x)$ satisfying $\sigma(x) \in U(x)$ where

$$U(x) = \{u \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\} \tag{6}$$

ensures forward invariance of $S$. Notably, the inequality in the definition of $U(x)$ is affine in $u$ and, therefore, can be included in convex optimization programs that compute a feedback controller $\sigma(x)$, possibly online at runtime. If such a feedback controller exists, then $h(x)$ is called a (classical) *Control Barrier Function (CBF)*.

### C. High-Order Control Barrier Functions

A common challenge in standard CBF-based control design is that, for many physically meaningful systems, $\nabla h(x)^T g(x)$ can be identically zero so that $U(x)$ becomes empty for some $x$. A possible solution is to use the theory of *High-Order Control Barrier Functions (HO-CBF)* that systematically constructs an alternative barrier function as follows: initialize $\psi_0(x) = h(x)$ and, as long as $\nabla \psi_i(x)^T g(x) \equiv 0$, iteratively set $\psi_{i+1}(x) = \nabla \psi_i(x)^T f(x) + \alpha_i(\psi_i(x))$ for some user-chosen Lipschitz functions $\alpha_i(\cdot)$. Suppose the process terminates after $r$ iterations. The resulting final $\psi_r(x)$ can often (*e.g.*, when the system has a well-defined uniform relative degree) be used as a CBF that guarantees forward invariance of $\cap_{i:i \leq r}\{x \mid \psi_i(x) \geq 0\}$, which is a subset of $S$.

## III. PIECEWISE-DEFINED BARRIER FUNCTIONS

Consider the system $\dot{x} = f(x, u)$ with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$ given by a feedback control law $u = \sigma(x)$. Suppose also a given nonsmooth function $h(x)$ and *safe set* $S = \{x \mid h(x) \geq 0\}$. The objective is to derive conditions ensuring that $S$ is forward invariant for the closed loop dynamics $f(x, \sigma(x))$ with the intention that such conditions can be used to design the controller $\sigma(x)$ for safety. This paper focuses on piecewise-defined barriers and piecewise continuous systems that naturally arise from piecewise-defined controllers. To motivate this approach, we first begin with an example and then formalize the insights from this example in the remainder of this section.

**Example 1.** *Consider the triple integrator system with dynamics*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \tag{7}$$

*which can be written as $\dot{x} = f(x) + g(x)u$ with $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$ for appropriate $f(x)$ and $g(x)$, and consider $h_0(x) = L - x_2 + |x_1|$ for some $L > 0$. At all $x$ such that $h_0(x)$ is differentiable, that is, all $x$ such that $x_1 \neq 0$, we have that $\nabla h_0(x)^T g(x) = 0$, preventing the application of existing nonsmooth CBF formulations. Consider instead $h(x)$ defined as*

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in X_1 \\ h_2(x) & \text{if } x \in X_2 \\ \max\{h_1(x), h_2(x)\} & \text{if } x \in \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2) \end{cases} \tag{8}$$

$$h_1(x) = -x_3 - x_2 + \beta(L - x_2 - x_1), \tag{9}$$

$$h_2(x) = -x_3 + x_2 + \beta(L - x_2 + x_1) \tag{10}$$

*for some $\beta > 0$ where $X_1 = \{x \in \mathbb{R}^3 : x_1 < 0\}$, $X_2 = \{x \in \mathbb{R}^3 : x_1 > 0\}$, and $\mathrm{cl}(X_1) \cap \mathrm{cl}(X_2) = \{x \in \mathbb{R}^3 : x_1 = 0\}$. As formalized in Section IV, $h(x)$ can be considered a generalized HO-CBF generated from the nonsmooth $h_0(x)$. Note that $h(x)$ is discontinuous when $x_1 = 0$. Consider that a Lipschitz feedback nominal controller $u_{nom}$ is given. The feedback safe controller $u = \sigma(x)$ is defined as*

$$\sigma(x) = \begin{cases} \min\{u_{nom}, \sigma_1(x)\} & \text{if } x \in X_1 \\ \min\{u_{nom}, \sigma_2(x)\} & \text{if } x \in X_2 \end{cases} \tag{11}$$

*where*

$$\sigma_1(x) = -x_3 + \beta(-x_3 - x_2) + \gamma h_1(x) \tag{12}$$

$$\sigma_2(x) = x_3 + \beta(-x_3 + x_2) + \gamma h_2(x) \tag{13}$$

*for some $\gamma > 0$. The result is a piecewise continuous system $\dot{x} = f(x) + g(x)\sigma(x)$ with unique Filippov solutions according to [18, Prop. 5].*

*We claim $S = \{x \mid h(x) \geq 0\}$ is forward invariant, which we show by establishing that system trajectories cannot cross $\partial S$, the boundary of $S$. By contradiction, suppose there exists an initial condition $x^0 \in \partial S$ such that $x(\tau) \notin S$ for all sufficiently small $\tau > 0$ for $x(t)$ initialized at $x^0$, $x(0) = x^0$. First consider the possibility that for small enough $\tau$, the trajectory remains within either $X_1$ or $X_2$, that is, for some fixed $i \in \{1, 2\}$, $x(t) \in X_i$ and $h(x(t)) = h_i(x(t))$ for all $t \in [0, \tau)$ for small enough $\tau$, and therefore we are reduced to the usual smooth setting. It is straightforward to verify that $\nabla h_i(x)^T (f(x) + g(x)\sigma_i(x)) \geq -\gamma h_i(x)$ for $i \in \{1, 2\}$, so that $h(x(t))$ remains nonnegative by standard CBF arguments as summarized in Section II-B, a contradiction.*

*This leaves the second possibility that $h(x^0) = h_i(x^0) \geq 0$ while $h(x(\tau)) = h_j(x(\tau)) < 0$ for all sufficiently small $\tau > 0$ for $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$, that is, the trajectory crosses from one part to the other, experiencing a discontinuous decrease in $h$. This requires $x_1^0 = 0$. Suppose first that $x_2^0 \geq 0$. Then $h_2(x^0) \geq h_1(x^0)$ so $h(x^0) = h_2(x^0) \geq 0$. But $\dot{x}_1 = x_2 \geq 0$, so the system cannot cross to $X_1$. A symmetric contradiction is obtained for $x_2^0 \leq 0$. We therefore conclude $S$ is forward invariant. Simulation results are shown in Figure 1, where we used a Proportional-Derivative nominal controller.* ■

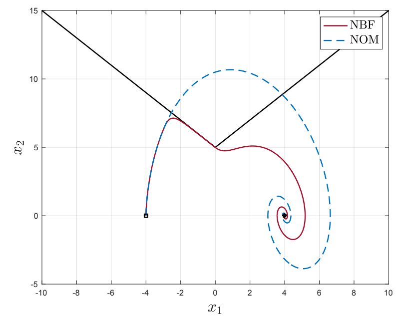


Fig. 1. Third order integrator chain trajectories. The safe region corresponding to $h_0 \geq 0$ is the outside of the inverted triangle. The initial position of the system is represented with a black square, and the final position with a black asterisk. The nominal controller without the barrier function filter violates safety (dashed blue line) whereas the trajectory obtained with the barrier function (solid red line) successfully avoids the unsafe region.

**Definition 1.** A function $h : \mathbb{R}^n \to \mathbb{R}$ is a *piecewise-defined barrier function candidate* on the open finite partition $\{X_i\}_{i \in \mathcal{I}}$ if there exists a collection of continuously differentiable $\{h_i(x)\}_{i \in \mathcal{I}}$ where each $h_i(x)$ is defined on $\mathrm{cl}(X_i)$ such that

1) $h(x) = \max_{i \text{ s.t. } x \in \mathrm{cl}(X_i)} \{h_i(x)\}$, and in particular $h(x) = h_i(x)$ for all $x \in X_i$ for all $i \in \mathcal{I}$;
2) For each $i \neq j$ with $\mathrm{cl}(X_i) \cap \mathrm{cl}(X_j) \neq \emptyset$, there exists a continuously differentiable $k_{i,j}(x)$ such that $X_i \subseteq \{x \mid k_{i,j} \geq 0\}$, $X_j \subseteq \{x \mid k_{i,j}(x) \leq 0\}$, and $X_i \cap X_j \subseteq \{x \mid k_{i,j}(x) = 0\}$. Without loss of generality, we may assume $k_{j,i}(x) = -k_{i,j}(x)$ for all $i, j$.

■

The second condition of Definition 1 implies that the boundaries between parts of the partition are characterized with continuously differentiable functions. Due to the definition of $h(x)$ in Definition 1, the safe set $S$ is closed and its boundary will be along the zero level sets of the constituent

$\{h_i(x)\}_{i\in\mathcal{I}}$ or along the boundaries between parts, leading immediately to Lemma 1 below.

**Lemma 1.** *For a piecewise-defined barrier function candidate with $S = \{x \mid h(x) \geq 0\}$, consider any $x \in \partial S$. Then, one of the following two conditions is true for $x$:*

1) *There exists $i$ such that $x \in X_i$ and $h_i(x) = 0$.*
2) *There exists $i$ such that $x \in \partial X_i$ and $h_i(x) \geq 0$.*

*Equivalently,*

$$\partial S \subseteq \bigcup_{i\in\mathcal{I}} \{x \mid h^i(x) = 0\} \cup \bigcup_{i\in\mathcal{I}} \{x \mid x \in \partial X_i,\ h^i(x) \geq 0\}, \tag{14}$$

*which implies*

$$\partial S \subseteq \bigcup_{i\in\mathcal{I}} \{x \mid h^i(x) = 0\} \cup \bigcup_{i\in\mathcal{I}} \partial X_i. \tag{15}$$

*Proof.* To prove Lemma 1 we need to look at two different cases. The first one corresponds to $h(x)$ being continuous at a given $x$, i.e., $x \in X_i$. Using the mean value theorem we know that $h(x)$ would have to have zero value before becoming negative and therefore according to the definition of $S$, $h(x) = 0$. The second case corresponds to an $x$ in the boundary between parts of the partition, i.e., $x \in \mathrm{cl}(X_i) \cap \mathrm{cl}(X_j)$. According to Definition 1, $h(x) = \max\limits_{i,j}\{h^i(x), h^j(x)\}$ which implies $h(x) \geq 0$. □

To facilitate the main result of this section, we introduce some further notation. First, given a piecewise-defined barrier function candidate $h(x)$ and corresponding open finite partition $\{X_i\}_{i\in\mathcal{I}}$, define the sets

$$\mathcal{O} = \cup_{i\in\mathcal{I}} X_i \tag{16}$$

$$\mathcal{B} = \mathbb{R}^n \backslash \mathcal{O} = \cup_{i\in\mathcal{I}} \partial X_i. \tag{17}$$

Define now the operator $\iota : \mathcal{O} \to \mathcal{I}$ as

$$\iota(x) = i \text{ if and only if } x \in X_i, \tag{18}$$

and finally, let $I : \mathcal{B} \to 2^{\mathcal{I}}$ be defined as

$$I(x) = \{i : x \in \mathrm{cl}(X_i)\}. \tag{19}$$

Equivalently, $I(x) = \{i : x \in \partial X_i\}$. For $x \in \mathcal{B}$, define the following three sets:

$$I_h^+(x) = \{i \in I(x) : h_i(x) > 0\} \tag{20}$$

$$I_h^-(x) = \{i \in I(x) : h_i(x) < 0\} \tag{21}$$

$$I_h^=(x) = \{i \in I(x) : h_i(x) = 0\}. \tag{22}$$

With this notation, Lemma 1 can be equivalently stated as: for any $x \in \partial S$, if $x \in \mathcal{O}$ then $h_{\iota(x)}(x) = 0$, otherwise, $x \in \mathcal{B}$ and $I_h^+(x) \cup I_h^=(x)$ is nonempty. We now impose the following mild technical assumption on the gradients of $k_{i,j}(x)$ and $h_i(x)$, which holds in many practical scenarios as shown in the examples in Sections III and IV.

**Assumption 1.** *For any $x \in \partial S \cap \mathcal{B}$,*

$$\nabla h_i(x) \neq 0 \ \forall i \in I_h^=(x) \tag{23}$$

$$\nabla k_{i,j}(x) \neq 0 \ \forall i, j \text{ s.t. } i \in I_h^+(x),\ j \in I_h^-(x). \tag{24}$$

*Moreover, the collection of vectors $\{\nabla k_{i,j}(x), \nabla h_i(x), \nabla h_j(x)\}$, noted as $\{v_i\}$, satisfies that for all $v_i$, $v_j \in \{v_i\}$ with $i \neq j$ there is no $\alpha > 0$ such that $v_i + \alpha v_j = 0$, i.e. $v_i$ and $v_j$ are not anti-parallel.*

We now state the main results of this paper. Theorem 1 enforces the vectorfield to point inside of the safe set $S$ at its boundary, while Theorem 2 uses Proposition 1 and Theorem 1 to establish sufficient conditions for forward invariance of $S$.

**Theorem 1.** *Given piecewise-defined barrier function candidate $h(x)$ and corresponding open finite partition $\{X_i\}_{i\in\mathcal{I}}$, let $S = \{x \mid h(x) \geq 0\}$. For any $x \in \partial S \cap \mathcal{B}$, let*

$$\mathcal{Z}_1(x) = \{z \mid \nabla h_i(x)^T z \geq 0 \ \forall i \in I_h^=(x)\} \tag{25}$$

$$\mathcal{Z}_2(x) = \tag{26}$$

$$\{z \mid \nabla k_{i,j}(x)^T z \geq 0 \ \forall i, j \text{ s.t. } i \in I_h^+(x),\ j \in I_h^-(x)\}.$$

*If Assumption 1 holds, then for all $x \in \partial S \cap \mathcal{B}$,*

$$\mathcal{Z}_1(x) \cap \mathcal{Z}_2(x) \subseteq \mathcal{T}_S(x). \tag{27}$$

*Proof.* Define the auxiliary sets $C_i = \{x \mid h_i(x) \geq 0\}$ and $K_{i,j} = \{x \mid k_{i,j}(x) \geq 0\}$ for all $i, j$. Fix $x \in \partial S \cap \mathcal{B}$. Assumption 1 ensures $\mathcal{T}_{C_i}(x) = \{v \mid \nabla h_i(x)^T v \geq 0\}$ and $\mathcal{T}_{K_{i,j}}(x) = \{v \mid \nabla k_{i,j}(x)^T v \geq 0\}$. There exists an open neighborhood $\mathcal{U}$ of $x$ such that

$$\mathcal{U} \cap \underbrace{\Bigg[\Big(\bigcap_{i\in\mathcal{I}^=(x)} C_i\Big) \cap \Big(\bigcap_{\substack{i,j \text{ s.t.} \\ i\in\mathcal{I}^+(x) \\ j\in\mathcal{I}\ \ (x)}} K_{i,j}\Big)\Bigg]}_{=:\Sigma} \subset (\mathcal{U} \cap S). \tag{28}$$

Since tangency is a local property, $\mathcal{T}_\Sigma(x) = \mathcal{T}_{\mathcal{U}\cap\Sigma}(x)$ and $\mathcal{T}_S(x) = \mathcal{T}_{\mathcal{U}\cap S}(x)$. As it holds that $(\mathcal{U} \cap \Sigma) \subset (\mathcal{U} \cap S)$, by [16, Pg. 99, Cond. a)] their tangent cones satisfy $\mathcal{T}_{\mathcal{U}\cap\Sigma}(x) \subset \mathcal{T}_{\mathcal{U}\cap S}(x)$. We thus obtain $\mathcal{T}_\Sigma(x) \subset \mathcal{T}_S(x)$. As the collection of vectors $\{\nabla k_{i,j}(x), \nabla h_i(x), \nabla h_j(x)\}$ satisfies Assumption 1, the sets $C_i$ and $K_{i,j}$ are *transversal* [16, Pg 99, Cond. c)] and therefore

$$\mathcal{T}_\Sigma(x) = \Big(\bigcap_{i\in\mathcal{I}^=(x)} \mathcal{T}_{C_i}(x)\Big) \cap \Big(\bigcap_{\substack{i,j \text{ s.t.} \\ i\in\mathcal{I}^+(x) \\ j\in\mathcal{I}\ \ (x)}} \mathcal{T}_{K_{i,j}}(x)\Big) \tag{29}$$

$$= \mathcal{Z}_1 \cap \mathcal{Z}_2 \subset \mathcal{T}_S(x). \tag{30}$$

□

**Theorem 2.** *Given a piecewise continuous system $\dot{x} = f(x)$ with unique Filippov solutions and piecewise-defined barrier function candidate $h(x)$, both with the same corresponding open finite partition $\{X_i\}_{i\in\mathcal{I}}$, satisfying Assumption 1. If there exists Lipschitz class-k functions $\alpha_i(\cdot)$ such that*

$$\nabla h_i(x)^T f(x) \geq -\alpha_i(h(x)) \ \forall i \in \mathcal{I} \text{ and all } x \in X_i \tag{31}$$

$$\nabla k_{i,j}(x)^T f(x) \geq 0 \quad \forall x \in \partial S \cap \mathcal{B} \text{ and } \forall i, j \text{ s.t.} \tag{32}$$

$$i \in I_h^+(x) \text{ and } j \in I_h^-(x),$$

*then $S = \{x \mid h(x) \geq 0\}$ is forward invariant.*

*Proof.* We want to prove that the intersection of the Filippov operator of $f(x)$ with the tangent cone of the safe set $S$ is not empty, i.e. $K[f(x)] \cap \mathcal{T}_S(x) \neq \emptyset$. There is two different cases of this proof: The first case corresponds to $x \in \partial S \cap \mathcal{O}$, what implies that $h_i = 0$. Due to the barrier condition $\nabla h_i(x)^T f(x) \geq 0$ holds, and Theorem 1 is satisfied for $z = f(x)$. The second case corresponds to $x \in \partial S \cap \mathcal{B}$. Equations (31) and (32) imply that $f(x) \subseteq \mathcal{T}_S(x)$, which according to Theorem 1 is true. Therefore, by definition of the Filippov operator [17] $f(x) \in K[f(x)]$ we have that $K[f(x)] \cap \mathcal{T}_S(x) \neq \emptyset$. According to Proposition 1 the set $S$ is forward invariant. □

**Example 2** (Example 1 continued)**.** *Given the system in* (7) *and the open finite partition $\{X_1, X_2\}$, define the boundary between $X_1$ and $X_2$ as $k_{2,1}(x) = -k_{1,2}(x) = x_1$. Using $h(x)$ as defined in* (8)–(10)*, Assumption 1 holds since $\nabla k_{1,2}(x)$, $\nabla h_1(x)$, and $\nabla h_2(x)$ are always nonzero and not mutually in opposite directions. Choosing $u(x)$ as in* (11)–(13) *satisfies* (31) *by construction. $u(x)$ is piecewise differentiable and given the dynamics of the system, [18, Prop. 5] guarantees that Filippov solutions exist and are unique. To verify* (32)*, consider $x \in \partial S \cap \mathcal{B}$ where $\mathcal{B} = \partial X_1 \cup \partial X_2 = \{x \mid x_1 = 0\}$. First, consider the case $x_2 \geq 0$, for which $h_1(x) \leq h_2(x)$. If $I_h^+(x)$ and $I_h^-(x)$ are nonempty, it must be that $I_h^+(x) = \{2\}$ and $I_h^-(x) = \{1\}$. Equation* (32) *gives $\nabla k_{2,1}(x)^T f(x) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T f(x) = x_2 \geq 0$, which holds when $x_2 \geq 0$ as assumed. A symmetric argument holds for $x_2 < 0$. As all conditions are met, Theorem 2 guarantees forward invariance of $S = \{x \mid h(x) \geq 0\}$.* ■

## IV. Discontinuous High-Order Control Barrier Functions

A salient feature of Example 2 is that (32) holds regardless of any chosen control, while (31) is enforced by choosing a controller to satisfy a classical CBF-type requirement. This is a consequence of $h(x)$ being built as a HO-CBF from a nonsmooth barrier function, as formalized in this section. Consider the system

$$\dot{x}_1 = f_1(x) \tag{33}$$

$$\dot{x}_2 = f_2(x) + g(x)u \tag{34}$$

for $f_1, f_2, g$ Lipschitz, $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Now consider a restricted class of piecewise-defined barrier functions on an open finite partition satisfying the following assumption.

**Assumption 2.** *For $x = (x_1, x_2) \in \mathbb{R}^n$ with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, the open finite partition $\{X_i\}_{i \in \mathcal{I}}$ of $\mathbb{R}^n$ satisfies the conditions of Assumption 1 and, furthermore, each $k_{i,j}(x)$ is a function of $x_1$ only, for which we instead write $k_{i,j}(x_1)$.*

Assumption 2 implies that each open finite partition $X_i \in \mathbb{R}^n$ is of the form $X_i^1 \times \mathbb{R}^{n_2}$ for appropriate $X_i^1 \subseteq \mathbb{R}^{n_1}$. Partitions of this form arise naturally when constructing high-order barrier functions from piecewise functions of $x_1$. Indeed, consider the system in (33)–(34) and let $\tilde{h}(x_1)$ be a piecewise-defined Lipschitz function of $x_1$ on the open finite partition $\{X_i^1\}_{i \in \mathcal{I}}$ of $\mathbb{R}^{n_1}$ so that, for all $i$, $\tilde{h}(x) = \tilde{h}_i(x)$ for continuously differentiable $\tilde{h}_i(x)$ for all $x \in X_i^1$. In general, this function cannot be used as a CBF for (33)–(34) because the relative degree between $u$ and $\tilde{h}(x_1)$ is greater than 1, i.e., $u$ affects $x_1$ only through its effect on $x_2$ in (34). An alternative barrier candidate $h(x)$ can be defined piecewise

$$h_i(x) = \nabla_{x_1} \tilde{h}_i(x_1)^T f_1(x) + \alpha_i(\tilde{h}_i(x_1)) \text{ for } x \in X_i \tag{35}$$

where $X_i = X_i^1 \times \mathbb{R}^{n_2}$ and defining $h(x) = \max_{i \in I(x)} h_i(x)$ for all $x \in \cup_{i \in \mathcal{I}} \partial X_i$. A collection $\{k_{i,j}(x_1)\}$ satisfying Assumption 1 for $\{X_i^1\}_{i \in \mathcal{I}}$ then satisfies Assumption 2 for $\{X_i\}_{i \in \mathcal{I}}$. The candidate barrier function (8)–(10) arises from such a construction as shown in Example 3. Under Assumption 2, we now specialize the conditions of Theorem 2 in the following corollary.

**Corollary 1.** *Given system* (33)–(34) *and a piecewise-defined barrier function candidate $h(x)$ defined on the open finite partition $\{X_i\}_{i \in \mathcal{I}}$ satisfying Assumptions 1 and 2. Suppose that*

$$\begin{aligned} \nabla_{x_1} k_{i,j}(x_1)^T f_1(x) \geq 0 \quad \forall i, j,\ \forall x = (x_1, x_2) \\ \text{s.t. } k_{i,j}(x_1) = 0, h_i(x) > 0, \text{ and } h_j(x) < 0. \end{aligned} \tag{36}$$

*If $u(x)$ is a piecewise-defined feedback control strategy that results in* (33)–(34) *being piecewise continuous on $\{X_i\}_{i \in \mathcal{I}}$ with unique Filippov solutions and satisfies*

$$\begin{aligned} \nabla_{x_1} h_i(x)^T f_1(x_1) + \nabla_{x_2} h_i(x)^T (f_2(x) + g(x)u(x)) \\ \geq -\alpha_i(h_i(x)) \ \forall i \in \mathcal{I},\ \forall x \in X_i, \end{aligned} \tag{37}$$

*then $S = \{x \mid h(x) \geq 0\}$ is forward invariant.*

The key feature of Corollary 1 is that (36) is independent of the control strategy $u(x)$ and therefore can be checked, e.g., offline. Then, standard CBF-based control synthesis methods can be used to enforce (37) at runtime.

**Example 3** (Example 1 revisited)**.** *Consider* (7) *and take $X_1^1 = \{(x_1, x_2) : x_1 < 0)\}$, $X_2^1 = \{(x_1, x_2) : x_1 > 0\}$, $\tilde{h}_1(x_1, x_2) = L - x_2 - x_1$ and $\tilde{h}_2(x_1, x_2) = L - x_2 + x_1$ for some $L > 0$, and $\tilde{h}(x_1, x_2)$. The resulting piecewise-defined barrier function candidate on $\{X_1^1, X_2^1\}$ is given by $\tilde{h}(x_1, x_2) = L - x_2 + |x_1|$. Taking $\alpha_1(s) = \alpha_2(s) = \beta s$ for some $\beta > 0$, and following the construction of* (35) *gives $h(x)$ as in* (8)–(10)*.* ■

**Example 4.** *The second-order unicycle is a five-dimensional, nonholonomic, nonlinear system with dynamics*

$$\begin{aligned} &(\dot{x}_1, \dot{x}_2) = (s\cos\theta, s\sin\theta) \\ &\dot{s} = u_a, \quad \dot{\theta} = \omega, \quad \dot{\omega} = u_\alpha, \end{aligned} \tag{38}$$

*where $u = (u_a, u_\alpha)$ is the two dimensional control input vector, and $x = (x_1, x_2, s, \theta, \omega)$ is the state vector with state variables: $(x_1, x_2)$ the unicycle center of mass position, $s$ its translational velocity, $\theta$ its heading angle, and $\omega$ its angular velocity. A barrier function that only depends on position $(x_1, x_2)$ cannot generally be used to construct a HO-CBF because the system does not have a well-defined*

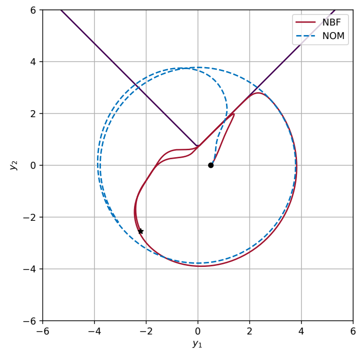


Fig. 2. Nonlinear second-order unicycle trajectories. The safe region corresponding to $\tilde{h} \geq 0$ is the outside of the inverted triangle. The initial position of the system is represented with a black dot, and the final position with a black asterisk. The nominal controller without the barrier function filter violates safety (dashed blue line) whereas the trajectory obtained with the barrier function (solid red line) successfully avoids the unsafe region.

*uniform relative degree. A standard solution for this system is to instead consider applying a barrier function to the point* $(y_1, y_2) = (x_1 + p\cos\theta, x_2 + p\sin\theta)$*, displaced from the center of mass by distance p in the direction of the heading θ [19]. Here, we take* $\tilde{h}(x_1, x_2, \theta) = L - (x_2 + p\sin\theta) + |x_1 + p\cos\theta|$. $\tilde{h}$ *is defined on the open finite partition* $X_1^1 = \{(x_1, x_2, \theta) : x_1 + p\cos\theta < 0\}$, $X_2^1 = \{(x_1, x_2, \theta) : x_1 + p\cos\theta > 0\}$*, with boundary function* $k_{2,1}(x) = -k_{1,2}(x) = x_1 + p\cos\theta$*. Following the construction in equation* (35) *using* $\alpha_1(s) = \alpha_2(s) = \beta s$ *for* $\beta > 0$*, we obtain*

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in X_1^1 \\ h_2(x) & \text{if } x \in X_2^1 \\ \max\{h_1(x), h_2(x)\} & \text{if } x \in \mathrm{cl}(X_1^1) \cap \mathrm{cl}(X_2^1), \end{cases} \tag{39}$$

$$\begin{aligned} h_1(x) = &- s\sin\theta - s\cos\theta - p\omega\cos\theta + p\omega\sin\theta \\ &+ \beta(L - x_2 - p\sin\theta - x_1 - p\cos\theta), \end{aligned} \tag{40}$$

$$\begin{aligned} h_2(x) = &- s\sin\theta + s\cos\theta - p\omega\cos\theta - p\omega\sin\theta \\ &+ \beta(L - x_2 - p\sin\theta + x_1 + p\cos\theta). \end{aligned} \tag{41}$$

*This barrier function satisfies* (36) *and we are able to find a piecewise-defined feedback control strategy satisfying* (37). *In particular, we take* $p = 0.5$*, and* $L = 0.7$ *and consider a nominal controller that tracks a circular trajectory. At each time instant, apply a CBF-based quadratic program to solve for the control input that is closest in Euclidean distance to the nominal input while satisfying the constraint in* (37)*, and apply this input as the feedback control strategy. An example of the resulting trajectory is shown in Figure 2 plotted in* $y_1$*-*$y_2$ *coordinates, where we observe that the trajectory is such that* $\tilde{h}$ *remains positive, as desired.* ■

## V. Conclusion

This paper presented a new method to certify forward invariance of a set defined as a level set of a piecewise-defined function. There are two key conditions to guarantee forward invariance: the first condition is a barrier function requirement on the interior of each piece of the state space partition, while the second condition requires the vectorfield to point inside of the safe set at the boundary between each pair of pieces of the partition. The second condition is independent of the control strategy and can be checked offline, whereas the first one can be used to design controllers in a CBF setting. These properties make our method particularly applicable for the discontinuous barrier functions that arise when building a high-order CBF from a piecewise-defined barrier candidate.

## References


[1] L. Wang, A. D. Ames, M. Egerstedt, Safety barrier certificates for collisions-free multirobot systems, IEEE Transactions on Robotics 33 (3) (2017) 661–674.

[2] G. Yang, B. Vang, Z. Serlin, C. Belta, R. Tron, Sampling-based motion planning via control barrier functions, in: Proceedings of the 2019 3rd International Conference on Automation, Control and Robots, 2019, pp. 22–29.

[3] A. D. Ames, J. W. Grizzle, P. Tabuada, Control barrier function based quadratic programs with application to adaptive cruise control, in: 53rd IEEE Conference on Decision and Control, IEEE, 2014, pp. 6271–6278.

[4] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, P. Tabuada, Control barrier functions: Theory and applications, in: 2019 18th European Control Conference (ECC), 2019, pp. 3420–3431.

[5] S. Prajna, A. Jadbabaie, G. J. Pappas, A framework for worst-case and stochastic safety verification using barrier certificates, IEEE Transactions on Automatic Control 52 (8) (2007) 1415–1428.

[6] W. Xiao, C. G. Cassandras, C. Belta, Safe Autonomy with Control Barrier Functions: Theory and Applications, Springer Nature, 2023.

[7] J. Usevitch, K. Garg, D. Panagou, Strong invariance using control barrier functions: A clarke tangent cone approach, in: 2020 59th IEEE Conference on Decision and Control (CDC), IEEE, 2020, pp. 2044–2049.

[8] P. Glotfelter, J. Cortés, M. Egerstedt, Nonsmooth barrier functions with applications to multi-robot systems, IEEE control systems letters 1 (2) (2017) 310–315.

[9] P. Glotfelter, J. Cortés, M. Egerstedt, A nonsmooth approach to controller synthesis for boolean specifications, IEEE Transactions on Automatic Control 66 (11) (2020) 5160–5174.

[10] P. Glotfelter, I. Buckley, M. Egerstedt, Hybrid nonsmooth barrier functions with applications to provably safe and composable collision avoidance for robotic systems, IEEE Robotics and Automation Letters 4 (2) (2019) 1303–1310.

[11] M. Maghenem, R. G. Sanfelice, Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions, Automatica 124 (2021) 109328.

[12] M. Maghenem, R. G. Sanfelice, Characterizations of safety in hybrid inclusions via barrier functions, in: Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control, 2019, pp. 109–118.

[13] W. Xiao, C. Belta, Control barrier functions for systems with high relative degree, in: 2019 IEEE 58th conference on decision and control (CDC), IEEE, 2019, pp. 474–479.

[14] X. Tan, W. S. Cortez, D. V. Dimarogonas, High-order barrier functions: Robustness, safety, and performance-critical control, IEEE Transactions on Automatic Control 67 (6) (2021) 3021–3028.

[15] W. Xiao, C. Belta, High-order control barrier functions, IEEE Transactions on Automatic Control 67 (7) (2021) 3655–3662.

[16] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, P. R. Wolenski, Nonsmooth analysis and control theory, Vol. 178, Springer Science & Business Media, 2008.

[17] B. Paden, S. Sastry, A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators, IEEE transactions on circuits and systems 34 (1) (1987) 73–82.

[18] J. Cortes, Discontinuous dynamical systems, IEEE Control systems magazine 28 (3) (2008) 36–73.

[19] W. Xiao, C. G. Cassandras, C. A. Belta, D. Rus, Control barrier functions for systems with multiple control inputs, in: 2022 American Control Conference (ACC), IEEE, 2022, pp. 2221–2226.