Satisfiability Bounds for ω-regular Properties in Interval-valued Markov Chains

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Abstract— **We derive an algorithm to compute satisfiability bounds for arbitrary ω-regular properties in an Interval-valued Markov Chain (IMC) interpreted in the adversarial sense. IMCs generalize regular Markov Chains by assigning a range of possible values to the transition probabilities between states. In particular, we expand the automata-based theory of ω-regular property verification in Markov Chains to apply it to IMCs. Any ω-regular property can be represented by a Deterministic Rabin Automata (DRA) with acceptance conditions expressed by Rabin pairs. Previous works on Markov Chains have shown that computing the probability of satisfying a given ω-regular property reduces to a reachability problem in the product between the Markov Chain and the corresponding DRA. We similarly define the notion of a product between an IMC and a DRA. Then, we show that in a product IMC, there exists a particular assignment of the transition values that generates a largest set of non-accepting states. Subsequently, we prove that a lower bound is found by solving a reachability problem in that refined version of the original product IMC. We derive a similar approach for computing a satisfiability upper bound in a product IMC with one Rabin pair. For product IMCs with more than one Rabin pair, we establish that computing a satisfiability upper bound is equivalent to lower-bounding the satisfiability of the complement of the original property. A search algorithm for finding the largest accepting and non-accepting sets of states in a product IMC is proposed. Finally, we demonstrate our findings in a case study.**

I. INTRODUCTION

Markov Chains have been extensively used as an intuitive yet powerful mathematical tool for modeling systems evolving through time in a stochastic fashion. They allow us to answer critical questions about the behavior of the underlying systems, often specified in terms of symbolic temporal logics, and derive appropriate control strategies [1] [2]. As a superset of *Linear Temporal Logic* (LTL), *ω-regular properties* are of particular interest to us due to their expressiveness. One can easily translate natural language inquiries such as *"Will the system never reach a bad state and visit a good state infinitely often?"* into well-defined regular expressions. A method for computing the probability of fulfilling any ω-regular property in Markov Chains is described in [3]. However, this derivation assumes that the probabilities of transition from state to state are known exactly.

Accessing the true probabilities of transitions might be impossible in practice and their values may only be approximated, e.g. from collected data. Furthermore, there has been a growing interest in abstractions of stochastic hybrid systems [4] [5], and the discretization of a stochastic continuous state space might sometimes result in a finite abstraction where the transitions between states cannot be expressed as a single number [6]. To account for this, Markov Chains are augmented into *Interval-valued Markov Chains (IMC)* where the probabilities of transition from state to state are given to lie within some interval [7] [8]. A direct consequence of this characteristic is that the probability of satisfying temporal properties in an IMC has to be formulated as an interval as well for all initial states.

Depending on the context in which they are utilized, IMCs give rise to two different semantic interpretations. One may view an IMC as an imperfect representation of a unique underlying Markov Chain whose transition bounds are not known exactly; this is called the *Uncertain Markov Chain* (UMC) interpretation of IMCs. On the other hand, IMCs can be interpreted in an adversarial sense where a new probability distribution consistent with the transition bounds is non-deterministically selected each time a state is visited. We refer to this as an *Interval Markov Decision Process* (IMDP).

In [9], the authors discuss the feasibility of the model-checking problem in both interpretations of IMCs and its computational complexity for ω-regular properties. Nevertheless, efficient algorithms for computing satisfiability bounds are not provided. Such bounds prove valuable in certain applications, such as the targeted state-space refinement of hybrid systems where IMCs naturally arise.

Satisfiability bounds were calculated in [10] for the Probabilistic Computation Tree Logic (PCTL) in IMDPs, but PCTL cannot express useful specifications such as liveness properties, i.e. the infinitely repeated occurence of an event. An automaton-based stochastic technique that asymptotically converges to lower and upper bounds for LTL formulas in UMCs was developed in [11]. To the best of our knowledge, a deterministic algorithm capable of finding satisfiability bounds for arbitrary ω-regular properties in IMDPs has not been presented in the literature and is the main contribution of this paper.

Our objective is to extend the automaton-based procedure in [3] to accommodate IMCs interpreted as IMDPs. All ω-regular properties can be converted into *Deterministic Rabin Automata* (DRA) whose acceptance conditions are described by sets of states grouped in pairs called *Rabin Pairs* [3]. Constructing the Cartesian product of a Markov Chain with a DRA enables to compute the probability that the stochastic evolution of the Markov Chain's state fulfills the property

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encoded in the DRA. In particular, it was shown that this probability is equal to that of reaching special sets of states called accepting *Bottom Strongly Connected Components* (BSCC) in the product Markov Chain. Unfortunately, such a straightforward procedure does not work in general for IMCs. Although a similar definition of the Cartesian product between an IMC and a DRA can be established, we observe that the set of accepting BSCCs depends on the assumed transition values in the resulting product IMC. The structure of a product IMC is indeed specifically determined by transitions which can either create or eliminate a path between two states, i.e. transitions with a zero probability lower bound and a non-zero upper bound.

Nonetheless, we first show that a particular instantiation on the transition values yields a largest set of so-called *non-accepting states*. Then, we show that computing a lower bound on the satisfiability of the property expressed by the DRA reduces to a reachability problem on the non-accepting states in the refined product IMC. If the underlying DRA only has one Rabin pair, we conversely prove that an upper bound is found by solving a reachability problem for a particular refinement of the product IMC that generates the most *accepting states*. In the case where the DRA possesses more than one Rabin pair, we show that an upper bound is calculated by lower-bounding the satisfiability for the complement property of the DRA. Furthermore, we describe an algorithm for finding the largest sets of non-accepting and accepting states in a product IMC. Lastly, we illustrate our algorithm in a case study.

In Section I, we introduce important concepts; in Section II, we formulate the problem; in Section III, we derive the main concepts used for bounding the satisfiability of ω-regular properties in IMCs and we present an algorithm for finding the largest sets of accepting and non-accepting states; finally, we present a case study in Section IV. Proofs and additional comments are omitted due to space constraints and are available in the extended version[1].

II. PRELIMINARIES

An *Interval-Valued Markov Chain (IMC)* [6] is a 5-tuple $\mathscr{I} = (Q, \check{T}, \hat{T}, \Pi, L)$ where:

- Q is a finite set of states,
- $\check{T} : Q \times Q \to [0,1]$ maps pairs of states to a lower transition bound so that $\check{T}_{Q_j \to Q_\ell} := \check{T}(Q_j, Q_\ell)$ denotes the lower bound of the transition probability from state Q_j to state Q_ℓ, and
- $\hat{T} : Q \times Q \to [0,1]$ maps pairs of states to an upper transition bound so that $\hat{T}_{Q_j \to Q_\ell} := \hat{T}(Q_j, Q_\ell)$ denotes the upper bound of the transition probability from state Q_j to state Q_ℓ,
- Π is a finite set of atomic propositions,
- $L : Q \to 2^{\Pi}$ is a labeling function that assigns a subset of Π to each state Q,

and $\check{T}$ and $\hat{T}$ satisfy $\check{T}(Q_j, Q_\ell) \leq \hat{T}(Q_j, Q_\ell)$ for all $Q_j, Q_\ell \in Q$ and

$$\sum_{Q_\ell \in Q} \check{T}(Q_j, Q_\ell) \leq 1 \leq \sum_{Q_\ell \in Q} \hat{T}(Q_j, Q_\ell) \tag{1}$$

for all $Q_j \in Q$.

A *Markov Chain* $\mathscr{M} = (Q, T, \Pi, L)$ is similarly defined with the difference that the transition probability function $T : Q \times Q \to [0,1]$ satisfies $0 \leq T(Q_j, Q_\ell) \leq 1$ for all $Q_j, Q_\ell \in Q$ and $\sum_{Q_\ell \in Q} T(Q_j, Q_\ell) = 1$ for all $Q_j \in Q$. At each discrete time step, a Markov Chain transitions from its current state Q_i to a state Q_j according to the probability distribution set by T. For any sequence of states $\pi = q_0 q_1 q_2 \ldots$ in $\mathscr{M}$, with $q_j \in Q$, q_0 is called an *initial state*.

A Markov Chain $\mathscr{M}$ is said to be *induced* by IMC $\mathscr{I}$ if for all $Q_j, Q_\ell \in Q$,

$$\check{T}(Q_j, Q_\ell) \leq T(Q_j, Q_\ell) \leq \hat{T}(Q_j, Q_\ell) \ . \tag{2}$$

An IMC $\mathscr{I}_2$ with transition functions $\check{T}_2$ and $\hat{T}_2$ is said to be *induced* by IMC $\mathscr{I}_1$ with transition functions $\check{T}_1$ and $\hat{T}_1$ if both $\mathscr{I}_1$ and $\mathscr{I}_2$ have the same Q, Π and L, and, for all $Q_j, Q_\ell \in Q$,

$$\check{T}_1(Q_j, Q_\ell) \leq \check{T}_2(Q_j, Q_\ell) \leq \hat{T}_2(Q_j, Q_\ell) \leq \hat{T}_1(Q_j, Q_\ell) \ . \tag{3}$$

Any Markov Chain induced by $\mathscr{I}_2$ is also induced by $\mathscr{I}_1$.

An IMC $\mathscr{I}$ is said to be interpreted as an *Interval Markov Decision Process* (IMDP) if, at each time step k, the external environment non-deterministically chooses a Markov chain $\mathscr{M}_k$ induced by $\mathscr{I}$ and the next transition occurs according to $\mathscr{M}_k$. A mapping ν from any finite path $\pi = q_0 q_1 \ldots q_k$ in $\mathscr{I}$ to a Markov Chain $\mathscr{M}_k$ is called an *adversary*. The set of all possible adversaries of $\mathscr{I}$ is denoted by $\nu_{\mathscr{I}}$.

An IMC $\mathscr{I}$ is said to be interpreted as an *Uncertain Markov Chain* (UMC) if the external environment non-deterministically chooses a single Markov chain $\mathscr{M}_0$ at $k = 0$ and the sequence of states $\pi = q_0 q_1 q_2 \ldots$ is determined by the transition probabilities in $\mathscr{M}_0$.

A *Deterministic Rabin Automaton (DRA)* [3] is a 5-tuple $\mathscr{A} = (S, \Sigma, \delta, s_0, Acc)$ where:

- S is a finite set of states,
- Σ is an alphabet,
- $\delta : Q \times \Sigma \to S$ is a transition function
- s_0 is an initial state
- $Acc \subseteq 2^S \times 2^S$. An element $(E_i, F_i) \in Acc$, with $E_i, F_i \subset S$, is called a *Rabin Pair*.

The probability of satisfying ω-regular property ϕ starting from initial state Q_i in IMC $\mathscr{I}$ under adversary ν is denoted by $\mathscr{P}_{\mathscr{I}[\nu]}(Q_i \models \phi)$. The greatest lower bound and the least upper bound probabilities of satisfying property ϕ starting from initial state Q_i in IMC $\mathscr{I}$ are denoted by $\check{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ and $\hat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ respectively. $\mathscr{P}_{\mathscr{M}}(Q_i \models \Diamond U)$ for $U \subseteq Q$ denotes the probability of eventually reaching set U from initial state Q_i in Markov Chain $\mathscr{M}$.

[1] The extended version can be found here: https://arxiv.org/abs/1809.06352

III. PROBLEM FORMULATION

Let $\mathscr{I}$ be an IMC interpreted as an IMDP with a set of atomic propositions Π, and let ϕ be an ω-regular property over alphabet Π (for formal definitions of ω-regular properties and alphabet, see [3]). Our goal is to find a method for calculating non-trivial bounds $\widecheck{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ and $\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ where, for any adversary $\nu \in \nu_{\mathscr{I}}$,

$$\widecheck{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi) \leq \mathscr{P}_{\mathscr{I}[\nu]}(Q_i \models \phi) \leq \widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)\,. \quad (4)$$

Our approach extends the work in [3] for the verification of regular Markov chains against ω-regular properties using automata-based methods. First, we generate a DRA $\mathscr{A}$ that recognizes the language induced by property ϕ. Such a DRA always exists and several algorithms can accomplish this task efficiently for a large subset of ω-regular expressions [12] [13]. Then, we construct the product $\mathscr{I} \otimes \mathscr{A}$.

Definition 1: Let $\mathscr{I} = (Q, \widecheck{T}, \widehat{T}, \Pi, L)$ be an Interval-valued Markov Chain and $\mathscr{A} = (S, 2^{\Pi}, \delta, s_0, Acc)$ be a Deterministic Rabin Automaton. The *product* $\mathscr{I} \otimes \mathscr{A} = (Q \times S, \widecheck{T}', \widehat{T}', Acc', L')$ is an Interval-valued Markov Chain where:

- $Q \times S$ is a set of states,
- $\widecheck{T}'_{\langle Q_j, s\rangle \rightarrow \langle Q_\ell, s'\rangle} = \begin{cases} \widecheck{T}'_{Q_j \rightarrow Q_\ell}, & \text{if } s' = \delta(s, L(Q_\ell)) \\ 0, & \text{otherwise} \end{cases}$
- $\widehat{T}'_{\langle Q_j, s\rangle \rightarrow \langle Q_\ell, s'\rangle} = \begin{cases} \widehat{T}'_{Q_j \rightarrow Q_\ell}, & \text{if } s' = \delta(s, L(Q_\ell)) \\ 0, & \text{otherwise} \end{cases}$
- $Acc' = \{E_1, E_2, \ldots, E_k, F_1, F_2, \ldots, F_k\}$ is a set of atomic propositions, where E_i and F_i are the sets in the Rabin pairs of Acc,
- $L' : Q \times S \rightarrow 2^{Acc'}$ such that $H \in L'(\langle Q_j, s\rangle)$ if and only if $s \in H$, for all $H \in Acc'$ and for all j.

A Markov Chain $\mathscr{M} \otimes \mathscr{A}$ induced by $\mathscr{I} \otimes \mathscr{A}$ is called a *product Markov Chain*.

The probability of satisfying ϕ from initial state Q_i in a Markov Chain equals that of reaching an accepting *Bottom Strongly Connected Component* (BSCC) from initial state $\langle Q_i, s_0\rangle$ in the product Markov Chain with $\mathscr{A}$ [3].

Definition 2: Given a Markov Chain $\mathscr{M}$ with states Q, a subset $B \subseteq Q$ is called a *Bottom Strongly Connected Component* (BSCC) of $\mathscr{M}$ if it satisfies the following conditions:

- B is strongly connected, that is, for each pair of states (q, t) in B, there exists a path fragment $q_0 q_1 \ldots q_n$ such that $T(q_i, q_{i+1}) > 0$ for $i = 0, 1, \ldots, n-1$, and $q_i \in B$ for $0 \leq i \leq n$ with $q_0 = q$ and $q_n = t$,
- no proper superset of B is strongly connected,
- $\forall s \in B$, $\Sigma_{t \in B} T(s, t) = 1$.

Definition 3: A Bottom Strongly Connected Component B of a product Markov Chain $\mathscr{M} \otimes \mathscr{A}$ is said to be *accepting*

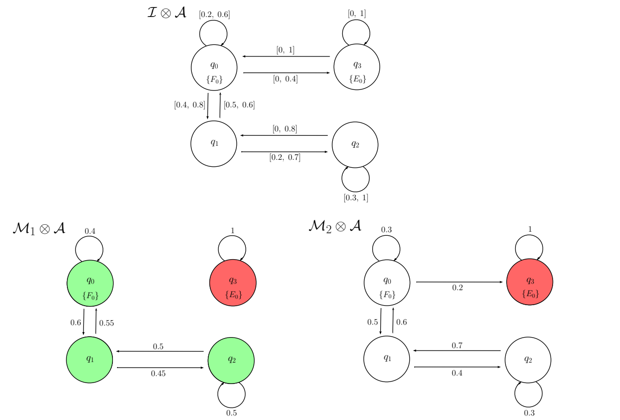

Fig. 1. Examples of two possible product Markov Chains $\mathscr{M}_1 \otimes \mathscr{A}$ and $\mathscr{M}_2 \otimes \mathscr{A}$ induced by the product IMC $\mathscr{I} \otimes \mathscr{A}$. The sets of accepting states U^A are shown in green whereas the sets of non-accepting states U^N appear in red. In $\mathscr{M}_1 \otimes \mathscr{A}$, $U^A = \{q_0, q_1, q_2\}$ and $U^N = \{q_3\}$; in $\mathscr{M}_2 \otimes \mathscr{A}$, $U^A = \{\emptyset\}$ and $U^N = \{q_3\}$.

if:

$$\exists i : \left(\exists \langle Q_j, s_\ell \rangle \in B \,.\, F_i \in L'(\langle Q_j, s_\ell \rangle) \right) \wedge \left(\forall \langle Q_j, s_\ell \rangle \in B \,.\, E_i \notin L'(\langle Q_j, s_\ell \rangle) \right). \quad (5)$$

In words, every state in a BSCC B is reachable from any state in B, and every state in B only transitions to another state in B. Moreover, B is accepting when at least one of its states maps to the accepting set of a Rabin pair, while no state in B maps to the non-accepting set of that same pair.

Definition 4: A state of $\mathscr{M} \otimes \mathscr{A}$ is *accepting* if it belongs to an accepting BSCC. The set of accepting states in $\mathscr{M} \otimes \mathscr{A}$ is denoted by $U^A_{\mathscr{M} \otimes \mathscr{A}}$; a state is *non-accepting* if it belongs to a BSCC that is not accepting. The set of non-accepting states in $\mathscr{M} \otimes \mathscr{A}$ is denoted by $U^N_{\mathscr{M} \otimes \mathscr{A}}$. We omit the subscripts when they are obvious from the context.

Each product Markov Chain $\mathscr{M} \otimes \mathscr{A}$ induced by $\mathscr{I} \otimes \mathscr{A}$ simulates the behavior of $\mathscr{I}$ under some adversary $\nu \in \nu_{\mathscr{I}}$. For any two states Q_j and Q_ℓ in $\mathscr{I}$ and some states s, s', s'' and s''' in $\mathscr{A}$, we allow $T_{\langle Q_j, s\rangle \rightarrow \langle Q_\ell, s'\rangle}$ and $T_{\langle Q_j, s''\rangle \rightarrow \langle Q_\ell, s'''\rangle}$ to assume different values in $\mathscr{M} \otimes \mathscr{A}$, which means that the transition probability between Q_j and Q_ℓ is permitted to change depending on the history of the path in $\mathscr{I}$.

Fact 1: [3] For any adversary $\nu \in \nu_{\mathscr{I}}$ in $\mathscr{I}$, it holds that $\mathscr{P}_{\mathscr{I}[\nu]}(Q_i \models \phi) = \mathscr{P}_{(\mathscr{M} \otimes \mathscr{A})_\nu}(\langle Q_i, s_0\rangle \models \Diamond U^A)$, where $(\mathscr{M} \otimes \mathscr{A})_\nu$ denotes the product Markov Chain induced by $\mathscr{I} \otimes \mathscr{A}$ corresponding to adversary ν.

It was shown in [14] that the IMDP and UMC interpretations yield identical results for reachability problems. Consequently, computing $\widecheck{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ and $\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ amounts to finding the product Markov Chains induced by $\mathscr{I} \otimes \mathscr{A}$ that respectively minimize and maximize the probability of reaching an accepting state. Such reachability problems in IMCs have been solved when the destination states are fixed for all induced Markov Chains [9] [10]. However, the set of accepting and non-accepting states may not be fixed in product IMCs and varies as a function of the assumed values for each transition. Specifically, U^A and U^N are determined by transitions whose lower bound is zero and upper bound non-zero, as in the example in Fig. 1.

Problem statement: "Given an IMC $\mathscr{I}$, an ω-regular property ϕ, and the DRA $\mathscr{A}$ corresponding to ϕ, find non-trivial bounds on the probability of reaching an accepting state from any initial state $\langle Q_i, s_0 \rangle$ in $\mathscr{I} \otimes \mathscr{A}$, and thereby find bounds on the probability of satisfying ϕ for any adversary ν in $\mathscr{I}$ and for any initial state Q_i."

We emphasize that this problem is non-trivial due the dependence of the set of accepting states on the assumed values for the transitions that can be "on" or "off".

IV. BOUNDING THE SATISFIABILITY OF ω-REGULAR PROPERTIES IN AN IMC

In [9], the authors discussed an algorithm for computing the probability bounds of reaching any fixed set of states in an IMC. We remarked in the previous section that, in general, the set of accepting and non-accepting states in a product IMC may depend on the assumed transition values. This is however not always the case.

Definition 5: A product IMC $\mathscr{I} \otimes \mathscr{A}$ is an *Accepting-Static Interval-Valued Markov Chain (ASIMC)* if for any two product Markov Chains $\mathscr{M}_1 \otimes \mathscr{A}$ and $\mathscr{M}_2 \otimes \mathscr{A}$ induced by $\mathscr{I} \otimes \mathscr{A}$, it holds that $(U^A)_{\mathscr{M}_1 \otimes \mathscr{A}} = (U^A)_{\mathscr{M}_2 \otimes \mathscr{A}}$.

Definition 6: A product IMC $\mathscr{I} \otimes \mathscr{A}$ is an *Non-Accepting-Static Interval-Valued Markov Chain (NASIMC)* if for any two product Markov Chains $\mathscr{M}_1 \otimes \mathscr{A}$ and $\mathscr{M}_2 \otimes \mathscr{A}$ induced by $\mathscr{I} \otimes \mathscr{A}$, it holds that $(U^N)_{\mathscr{M}_1 \otimes \mathscr{A}} = (U^N)_{\mathscr{M}_2 \otimes \mathscr{A}}$.

In ASIMCs (NASIMCs), the set of accepting (non-accepting) states remains the same for all induced product Markov Chains. Therefore, we can apply the standard reachability techniques in [9] to compute bounds on $\mathscr{P}_{\mathscr{I} \otimes \mathscr{A}}(\langle Q_i, s_0 \rangle \models \Diamond U^A)$ or $\mathscr{P}_{\mathscr{I} \otimes \mathscr{A}}(\langle Q_i, s_0 \rangle \models \Diamond U^N)$ in such product IMCs.

Notice that any product IMC $\mathscr{I} \otimes \mathscr{A}$ induces at most $(|Q| \cdot |S|)^{|Q| \cdot |S|}$ combinations of "on" and "off" transitions. A computationally inefficient technique is to bound the satisfiability of ϕ for every such combinations. In this section, we develop a more efficient method for solving this problem. First, we prove that all product IMCs induce a worst-case NASIMCs containing the largest set of non-accepting states and in which the probability of reaching an accepting BSCC is minimized from any initial state. Then, we show that the converse best-case ASIMCs are always induced by product IMCs with one Rabin pair, and the probability of reaching an accepting BSCC is maximized from any initial state in those ASIMCs. If the DRA for ϕ possesses more than one Rabin pair, we determine an upper bound by computing a lower bound on the satisfiability for the complement of ϕ. Finally, we suggest a search algorithm for efficiently finding the largest sets of accepting and non-accepting states.

A. Lower Bound Computation

Lemma 1: [3] For any infinite sequence of states $\pi = q_0 q_1 q_2 \dots$ in a Markov Chain, there exists an index $i \geq 0$ such that q_i belongs to a BSCC.

Lemma 2: Let $\mathscr{I} \otimes \mathscr{A}$ be a product IMC. Let $(\mathscr{I} \otimes \mathscr{A})_1$ and $(\mathscr{I} \otimes \mathscr{A})_2$ be two product NASIMCs induced by $\mathscr{I} \otimes \mathscr{A}$ with sets of non-accepting states U_1^N and U_2^N respectively. There exists an NASIMC $(\mathscr{I} \otimes \mathscr{A})_3$ induced by $\mathscr{I} \otimes \mathscr{A}$ with non-accepting states U_3^N and such that $\left(U_1^N \cup U_2^N\right) \subseteq U_3^N$.

Corollary 1: Let $\mathscr{I} \otimes \mathscr{A}$ be a product IMC. There exists a NASIMC induced by $\mathscr{I} \otimes \mathscr{A}$ with a set of non-accepting states U_ℓ^N and such that $U_i^N \subseteq U_\ell^N$, where U_i^N is the set of non-accepting states for any product Markov Chain $(\mathscr{M} \otimes \mathscr{A})_i$ induced by $\mathscr{I} \otimes \mathscr{A}$.

Remark 1: Let $[\mathscr{I} \otimes \mathscr{A}]^N$ be the set of all NASIMCs induced by $\mathscr{I} \otimes \mathscr{A}$ producing non-accepting states U_ℓ^N from Corollary 1. We denote the transition bounds functions of $\mathscr{I} \otimes \mathscr{A}$ by $\breve{T}$ and $\widehat{T}$. There exists a non-empty set of NASIMCs $[\mathscr{I} \otimes \mathscr{A}]_\ell^N \subseteq [\mathscr{I} \otimes \mathscr{A}]^N$ such that, for all $(\mathscr{I} \otimes \mathscr{A})_\ell \in [\mathscr{I} \otimes \mathscr{A}]_\ell^N$ with transition functions $\breve{T}_\ell$ and $\widehat{T}_\ell$, $\breve{T}_\ell(\ \langle Q_i, s_i \rangle, \langle Q_j, s_j \rangle\ ) = \breve{T}(\ \langle Q_i, s_i \rangle, \langle Q_j, s_j \rangle\ )$ and $\widehat{T}_\ell(\ \langle Q_i, s_i \rangle, \langle Q_j, s_j \rangle\ ) = \widehat{T}(\ \langle Q_i, s_i \rangle, \langle Q_j, s_j \rangle\ )$ for all $\langle Q_i, s_i \rangle \notin U_\ell^N$ and all $\langle Q_j, s_j \rangle \in Q \times S$.

Lemma 3: Let $\mathscr{I} \otimes \mathscr{A}$ be a product IMC. Consider two sets of non-accepting states U_1^N and U_2^N which can be induced by $\mathscr{I} \otimes \mathscr{A}$ and such that $U_2^N \subseteq U_1^N$. For any NASIMC $(\mathscr{I} \otimes \mathscr{A})_2$ with non-accepting states U_2^N induced by $\mathscr{I} \otimes \mathscr{A}$, there exists a NASIMC $(\mathscr{I} \otimes \mathscr{A})_1$ with non-accepting states U_1^N induced by $\mathscr{I} \otimes \mathscr{A}$ such that, for any initial state $\langle Q_i, s_0 \rangle$,

$$\widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_1}(\langle Q_i, s_0 \rangle \models \Diamond U_1^N) \geq \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_2}(\langle Q_i, s_0 \rangle \models \Diamond U_2^N) \ . \tag{6}$$

Lemma 4: Let $\mathscr{I} \otimes \mathscr{A}$ be a product IMC. Let $[\mathscr{I} \otimes \mathscr{A}]^N$ and $[\mathscr{I} \otimes \mathscr{A}]_\ell^N$ be the sets as defined in Remark 1. For any NASIMC $(\mathscr{I} \otimes \mathscr{A})' \in [\mathscr{I} \otimes \mathscr{A}]^N$, any NASIMC $(\mathscr{I} \otimes \mathscr{A})_\ell \in [\mathscr{I} \otimes \mathscr{A}]_\ell^N$ and any initial state $\langle Q_i, s_0 \rangle$,

$$\widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_\ell}(\langle Q_i, s_0 \rangle \models \Diamond U_\ell^N) \geq \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})'}(\langle Q_i, s_0 \rangle \models \Diamond U_\ell^N) \ .$$

Theorem 1: Let $\mathscr{I}$ be an IMC and $\mathscr{A}$ be a DRA corresponding to the ω-regular property ϕ. Let $(\mathscr{I} \otimes \mathscr{A})_\ell$ and $(\mathscr{I} \otimes \mathscr{A})'$ be any two NASIMCs from the set $[\mathscr{I} \otimes \mathscr{A}]_\ell^N$ defined in Remark 1. It holds that $\widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_\ell}(\langle Q_i, s_0 \rangle \models \Diamond U_\ell^N) = \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})'}(\langle Q_i, s_0 \rangle \models \Diamond U_\ell^N)$ and, for any state $Q_i \in \mathscr{I}$,

$$\breve{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi) = 1 - \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_\ell}(\langle Q_i, s_0 \rangle \models \Diamond U_\ell^N) \ . \tag{7}$$

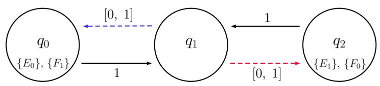

Fig. 2. Example where the analogous version of Lemma 2 for accepting states does not hold. Setting the blue transition to 1 and the red transition to 0 makes q_0 and q_1 accepting. Conversely, setting the blue transition to 1 and the red transition to 0 renders q_1 and q_2 accepting. Nonetheless, no assignment makes q_0, q_1 and q_2 accepting at the same time.

B. Upper Bound Computation

Due to the acceptance condition of a DRA, the analogous version of Lemma 2 for accepting states does not always hold true, as in Fig. 2. We thus treat IMCs with one Rabin pair and those with more than one Rabin pair separately.

1) Product IMC with one Rabin pair:

We denote by U_u^A the largest set of accepting states induced by $\mathscr{I} \otimes \mathscr{A}$. We define the set of best case ASIMCs $[\mathscr{I} \otimes \mathscr{A}]_u^A$ analogously to the set $[\mathscr{I} \otimes \mathscr{A}]_l^N$ for NASIMCs.

Theorem 2: Let $\mathscr{I}$ be an IMC and $\mathscr{A}$ be a DRA with one Rabin pair corresponding to the ω-regular property ϕ. Let $(\mathscr{I} \otimes \mathscr{A})_u$ and $(\mathscr{I} \otimes \mathscr{A})'$ be any two ASIMCs from the set $[\mathscr{I} \otimes \mathscr{A}]_u^A$. It holds that $\widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_u}(\langle Q_i, s_0\rangle \models \Diamond U_u^A) = \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})'}(\langle Q_i, s_0\rangle \models \Diamond U_u^A)$ and, for any state $Q_i \in \mathscr{I}$,

$$\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi) = \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \mathscr{A})_u}(\langle Q_i, s_0\rangle \models \Diamond U_u^A) \ . \tag{8}$$

2) Product IMC with more than one Rabin pairs:

We observed that product IMCs with more than one Rabin pair don't necessarily induce a unique largest set of accepting states. Instead, we exploit the fact that ω-regular expressions are closed under complementation [15].

Theorem 3: Let $\mathscr{I}$ be an IMC and $\overline{\mathscr{A}}$ be a DRA corresponding to $\neg\phi$, the complement of the ω-regular property ϕ. $(\mathscr{I} \otimes \overline{\mathscr{A}})_\ell$ is defined analogously as in Theorem 1. For any state $Q_i \in \mathscr{I}$,

$$\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi) = \widehat{\mathscr{P}}_{(\mathscr{I} \otimes \overline{\mathscr{A}})_\ell}(\langle Q_i, s_0\rangle \models \Diamond U_\ell^N) \ . \tag{9}$$

C. Search Algorithm

We design a search algorithm for finding U_l^N and U_u^A in a product IMC $\mathscr{I} \otimes \mathscr{A}$:

- Generate a directed graph $G(V,E)$ with a vertex for each state in $\mathscr{I} \otimes \mathscr{A}$. An edge links two states $\langle Q_i, s_i\rangle$ and $\langle Q_i', s_i'\rangle$ if $\widehat{T}_{\langle Q_i, s_i\rangle \to \langle Q_i', s_i'\rangle} > 0$.
- Find all *strongly connected components* (SCC) in G, e.g using Kosaraju's algorithm [16], and list them in C.
- For all SCC $C^j \in C$, check whether it contains a *leaky* state: a state $\langle Q_i, s_i\rangle \in C^j$ is leaky if, for some state $\langle Q_i', s_i'\rangle \notin C^j$, $\widecheck{T}_{\langle Q_i, s_i\rangle \to \langle Q_i', s_i'\rangle} > 0$ or if $\Sigma_{\langle Q_i', s_i'\rangle \in C^j} \widehat{T}_{\langle Q_i, s_i\rangle \to \langle Q_i', s_i'\rangle} < 1$ (that is, $\langle Q_i, s_i\rangle$ has a non-zero probability of transitioning outside of C_j for all refinement of $\mathscr{I} \otimes \mathscr{A}$).

Algorithm 1: Probability bounds computation for ω-regular properties in IMCs

Input : Interval-valued Markov Chain $\mathscr{I}$, ω-regular property ϕ.
Output: Lower and upper bound probabilities of satisfying ϕ in $\mathscr{I}$, $\widecheck{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ and $\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$, for all initial states Q_i.

Construct a DRA $\mathscr{A}$ corresponding to ϕ;
Generate the product $\mathscr{I} \otimes \mathscr{A}$;
Find the largest set of non-accepting states U_ℓ^N in $\mathscr{I} \otimes \mathscr{A}$ according to our search algorithm;
Compute $\widecheck{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ for all Q_i using (8) and the reachability algorithm in [9];

if $|Acc| = 1$ **then**
  Find the largest set of non-accepting states U_u^A in $\mathscr{I} \otimes \mathscr{A}$ according to our search algorithm;
  Compute $\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ for all Q_i using (9) and the reachability algorithm in [9];
else
  Construct a DRA $\overline{\mathscr{A}}$ corresponding to $\overline{\phi}$, the complement of ϕ;
  Generate the product $\mathscr{I} \otimes \overline{\mathscr{A}}$;
  Find the largest set of non-accepting states U_ℓ^N in $\mathscr{I} \otimes \overline{\mathscr{A}}$ according to our search algorithm;
  Compute $\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$ for all Q_i using (10) and the reachability algorithm in [9];
end

return $\widecheck{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$, $\widehat{\mathscr{P}}_{\mathscr{I}}(Q_i \models \phi)$

- If a state $\langle Q_i, s_i\rangle \in C^j$ is leaky, it cannot belong to a BSCC. Find all states in C^j whose transition to a leaky state cannot be "turned off" as above. These states are also leaky. Repeat for all leaky states in C^j.
- In the subgraph G^j induced by C^j, remove all edges from non-leaky to leaky states. Find all SCCs in G^j and add them to C.
- If C^j has no leaky state, C^j is a BSCC. For all states in C^j, check if it maps to some accepting set F_i. If not, C^j is a non-accepting BSCC. Otherwise, we treat two different cases for U_l^N and U_u^A.
- Search for U_l^N: For all such F_i's, check whether some state in C^j maps to the corresponding non-accepting set E_i. If this is the case for all such F_i's, C^j is a non-accepting BSCC. Otherwise, the unmatched F_i states cannot belong to a non-accepting BSCC. Treat them as leaky and follow the same procedure as before for eliminating leaky states. Add the new SCCs to C.
- Search for U_u^A: Check whether some state in C^j maps to E_1. If not, C^j is accepting and is U_u^A. Otherwise, treat the states mapping to E_1 as leaky and follow the same procedure as before for eliminating leaky states. Add the new SCCs to C.

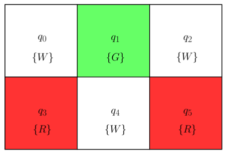

Fig. 3. A grid representation of the 6 states the agent can be in.

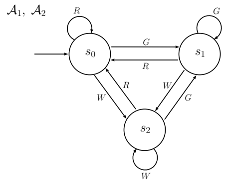

Fig. 4. A state diagram for automata $\mathscr{A}_1$ and $\mathscr{A}_2$ corresponding to properties ϕ_1 and ϕ_2 respectively. While their set of states and transition function are identical, they display different acceptance conditions: for $\mathscr{A}_1$, $Acc = \{(\{s_0\}, \{s_1\})\}$ whereas for $\mathscr{A}_2$, $Acc = \{(\emptyset, \{s_1\}),\ (\{s_0, s_1\}, \{s_2\})\}$.

	q_0	q_1	q_2	q_3	q_4	q_5
Lower bound for ϕ_1	0	0	0	0	0	0
Upper bound for ϕ_1	0	0	0	0	0	0
Lower bound for ϕ_2	0.274	0.368	1	0	1	0.684
Upper bound for ϕ_2	0.7	1	1	0	1	1

TABLE I

BOUNDS FOR THE SATISFIABILITY OF ϕ_1 AND ϕ_2

For any state belonging to these sets, the reachability probabilities are trivially 1. For all other states, the maximum probabilities of reaching U_u^A or U_l^N can be computed using the algorithm in [9] since these sets are now fixed. Algorithm 1 summarizes the entire procedure for bounding the satisfiability of ω-regular properties in IMCs.

V. CASE STUDY

Our system of interest is an agent moving stochastically on a grid shown in Fig. 3. The grid is divided into 6 different states the agent can visit. The system evolves in discrete-time: at each time t, the agent makes a transition from its current state to a new state according to some probability distribution. The latter depends only on the current state of the agent, i.e. the system satisfies the Markov property.

However, the transition probabilities are not known exactly and an IMC representation of the system is constructed:

$\check{T}$	q_0	q_1	q_2	q_3	q_4	q_5
q_0	0	0.2	0	0.3	0.2	0
q_1	0	0.05	0.25	0	0.1	0
q_2	0	0	0	0	1	0
q_3	0	0	0	1	0	0
q_4	0	0	1	0	0	0
q_5	0	0.3	0.2	0	0.2	0

$\widehat{T}$	q_0	q_1	q_2	q_3	q_4	q_5
q_0	0	0.5	0.3	0.6	0.5	0
q_1	0.2	0.8	0.6	0.8	0.7	0.5
q_2	0	0	0	0	1	0
q_3	0	0	0	1	0	0
q_4	0	0	1	0	0	0
q_5	0	0.5	0.5	0	0.3	0

Each state is labeled as follows: $L(q_0) = L(q_2) = L(q_4) = \{W\}$, $L(q_1) = \{G\}$ and $L(q_3) = L(q_5) = \{R\}$. We aim to bound the probability of satisfying ω-regular properties ϕ_1 and ϕ_2, represented by automata $\mathscr{A}_1$ and $\mathscr{A}_2$ in Fig. 4, from every initial state q. In natural language, these properties respectively translate to *"The agent visits a green state infinitely many times while visiting a red state finitely many times."* and *"The agent shall visit a red state infinitely many times only if it visits a green state infinitely many times."* Note that $\mathscr{A}_2$ has 2 Rabin pairs. According to Theorem 3, we thus have to construct the automata for the complement of ϕ_2. ϕ_2 is expressed in LTL as $\phi_2 = \Box\Diamond G \vee \Diamond\Box W$ which, when complemented, becomes $\neg\phi_2 = \Diamond\Box\overline{G} \wedge \Box\Diamond\overline{W}$. Then, we convert $\neg\phi_2$ to a DRA with one of the existing LTL-to-ω-automata translation tools [17]. Bounds for ϕ_1 and ϕ_2 are computed using Algorithm 1 and are shown in Table 1.

VI. CONCLUSIONS

We derived an automaton-based technique for bounding the probability of satisfying any ω-regular property in an IMC viewed as an IMDP. We demonstrated its application through a case study. In future works, we will seek to exploit the mechanisms unveiled in this paper and apply them to Bounded-parameter Markov Decision Processes, the controllable counterparts of IMCs, e.g. to minimize or maximize the probability of satisfying some specification.

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