Bounding the State Covariance Matrix for Switched Linear Systems with Noise

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***Abstract*— This paper studies the infinite-time behavior of switched linear systems in the presence of additive noise. In particular, we show that the propagation of the state covariance matrix can be described by a linear affine system and therefore classified by an invariant region of the covariance space. An algorithm is presented for bounding the state covariance matrix with a suitable hyper-ellipsoid in the dimension of the covariance space; we form this algorithm using a Kronecker algebra-based derivation.**

I. Introduction

Safety guarantees for real-world systems are often derived from switched dynamical system models [1]. In some instances, this is due to the inherent modal nature of the system dynamics [2], [3], but the class of systems which can be abstracted with switched models is quite broad. Certain classes of nonlinear systems, for example, can be modeled with switching dynamics, as can systems which experience time delays [4], [5]. As a result, academic work on the stability of switching systems can be distilled into many overlapping categories. Many works explore stability guarantees and control synthesis techniques arising from conditions on the system structure and switching scheme. Some deal with systems that switch deterministically at boundaries in the state space [6], [7]. Others provide conditions for controllability and develop feedback control strategies for switched-linear systems [8], [9]. Feedback control strategies have been developed for nonlinear systems with stochastic switching [10] and for systems with time delays [11], [12].

This work explores systems in which the switching sequence is arbitrary. In [13], [14] stability guarantees follow from the construction of a piecewise quadratic Lyapunov function for such systems. In many studies of switched systems, a Linear Matrix Inequality (LMI) is formulated in order to solve numerically for a positive definite matrix or set of such matrices which parameterize a single quadratic Lyapunov function for the switched system [15]. Here, we consider a switched linear system with additive Gaussian noise and ultimately formulate an LMI which can be solved to compute an ellipsoidal bound for the covariance matrix of the system state.

The covariance matrix of this switched linear system with noise does not hold a single value. We show that it evolves as the state of a related "augmented" system. As shown in [16], for the case of a continuous-time switching system without random noise, this augmented system is stable if and only if the original arbitrarily switching system is stable. In the present case, we are not concerned with asymptotic stability but rather that the state covariance is bounded. The state covariance evolves according to a switching linear affine difference equation, and the state of such a system often approaches an attractor set [17] rather than an equilibrium point. The authors in [18] find a sufficient condition for which the set of possible covariances form a fractal set in a Kalman filtering problem. Another related study seeks to find an optimal control scheme to control the covariance matrix for a stochastic discrete-time linear time-varying system, steering it from an initial probabilistic distribution to a desired one [19]. In this research, we present an algorithm for computing an ellipsoidal bound on the set of possible covariance matrices for the switching system with noise, which is equivalent to a bound on the minimal attractor set of the augmented system.

This paper is organized as follows. We define the system under study, and then proceed to look at the case of a randomly switching affine system. In other words, the state propagation is subject to multiple sets of dynamics, each of which may contain a different equilibrium point. A stability guarantee or, at the very least, a bound on the minimal attractor set for such a system, constitutes a preliminary development of this paper. Then, we show that this randomly switching affine system is of the same form as a randomly switching system that propagates a covariance matrix (as the system state) for a randomly switching system with noise. Numerical examples demonstrate how the ellipsoid determined through the matrix inequality formulation solves the guaranteed bound problem.

II. Bounding the Infinite-Time Behavior of Randomly Switching Affine Systems

This work considers discrete-time dynamical systems of the form

$$\begin{aligned} x(k+1) &= A(k)x(k) + w(k) \\ w(k) &\sim \mathcal{N}(\mu(k),\, \Sigma(k)) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ denotes the system state, and $w(k) \in \mathbb{R}^n$ denotes an additive Gaussian noise term with mean $\mu(k) \in$

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$\mathbb{R}^n$ and noise covariance matrix $\Sigma(k) \in \mathbb{R}^{n\times n}$. We further assume that the tuple $(A(k),\, \mu(k),\, \Sigma(k)) \in \mathcal{C} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ is chosen at each time step from a finite set of system parameters:

$$\mathcal{C} \triangleq \Big\{(A_1,\, \mu_1,\, \Sigma_1),\, \cdots,\, (A_N,\, \mu_N,\, \Sigma_N)\Big\}.$$

Importantly, we make no assumptions on the stochastic properties of $(A(k),\, \mu(k),\, \Sigma(k))$ within $\mathcal{C}$.

We are specifically interested in classifying the infinite-time behavior of the system (1) under arbitrary switching and identifying any invariant regions of the statespace, should they exist. To that end, we first consider an affine reduction of (1). After deriving a general framework for computing invariant regions for switched affine systems, we return to the initial stochastic setting of (1).

A. Mathematical Preliminaries

Consider the discrete-time linear affine system

$$x(k+1) = A(k)x(k) + w(k) \tag{2}$$

where $x(k) \in \mathbb{R}^n$ denotes the system state and the pair $(A(k),\, w(k)) \in \mathcal{C}'$ is chosen arbitrarily at each timestep from a finite set of system parameters:

$$\mathcal{C}' \triangleq \Big\{(A_1,\, w_1),\, \cdots,\, (A_N,\, w_N)\Big\}.$$

We further assume that for all $i \in \{1,\, \cdots,\, N\}$ the eigenvalues of A_i have a magnitude less than one; in this case, each subsystem

$$\begin{gathered} x(k+1) = A_i x(k) + w_i \\ i \in \{1,\, \cdots,\, N\} \end{gathered} \tag{3}$$

converges globally to an equilibrium point $x_{\text{eq},i}$ given by

$$x_{\text{eq},i} = (I_n - A_i)^{-1} w_i, \tag{4}$$

where $I_n \in \mathbb{R}^{n\times n}$ denotes the $n \times n$ identity matrix.

The system (2) does not have an equilibrium point in general, unless $x_{\text{eq},i} = x_{\text{eq},j}$ for all $i,\, j \in \{1,\, \cdots,\, N\}$. In this specific case, however, one cannot assume the stability properties of the equilibrium under arbitrary switching; that is, the presence of a unique equilibrium point for the switched system (2) does not guarantee global convergence, even in the case that each switched mode of (3) is itself stable. For this reason, it is preferable to instead classify the infinite-time behavior of (2) with invariant regions of the state space.

B. The Attractor Set

A related notion to equilibrium is that of the attractor set, which is an invariant region of the state space to which all initial system states converge.

Definition 1 (Attractor Set)**.** We use the symbol $d(x, \mathcal{S})$ to denote the Euclidean distance between the vector $x \in \mathbb{R}^n$ and a set $\mathcal{S} \subset \mathbb{R}^n$, which we define by

$$d(x, \mathcal{S}) = \min_{y\in\mathcal{S}} \|y - x\|$$

where $\|\cdot\|$ is the Euclidean norm. A closed set $\mathcal{A} \subset \mathbb{R}^n$ is a global *attractor* for (2) if for all switching sequences $(A(0), w(0)), (A(1), w(1)), \ldots$ with $(A(k), w(k)) \in \mathcal{C}'$ for all k, and for all initial conditions $x(0) \in \mathbb{R}^n$, the resulting state trajectory $x(k)$ satisfies

$$\lim_{k\to\infty} d(x(k), \mathcal{A}) = 0.$$

An attractor is *minimal* if no strict subset is also an attractor, and we use $\mathcal{A}_m$ to denote the global minimal attractor set of (2).

The intersection of any collection of attractors is itself an attractor. It thus follows that the minimal attractor for a given system is unique. Little is known in general about the shape or size of $\mathcal{A}_m$. It often features a fractal structure, as shown in a numerical example provided at the end of this section.

We aim to compute an invariant, hyper-ellipsoidal outer approximation of $\mathcal{A}_m$, which will take the form

$$\mathcal{E}_{P,x_c} = \big\{x \in \mathbb{R}^n \mid (x - x_c)^T P (x - x_c) \le 1\big\} \subset \mathbb{R}^n$$

where $x_c \in R^n$ denotes the centroid of the ellipse and where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix which encodes the orientation and scaling. We choose to focus on ellipsoidal outer-approximations since the search for the ellipsoid parameter P can be easily formulated as a semidefinite programming (SDP) problem. The following proposition specifies a requirement on the time-evolution of the ellipsoidal set which ensures that the ellipsoid is an outer-approximation of $\mathcal{A}_m$. The proposition is similar to the LaSalle-like invariance theorem in [20] but with relaxed assumptions since only systems of the form (2) are considered, and only an outer-approximation of an invariant set is sought rather than a proof of asymptotic stability.

Proposition 1. *Define*

$$E(x) \triangleq (x - x_c)^T P (x - x_c), \tag{5}$$

for some vector $x_c \in \mathbb{R}^n$ and some symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$. Additionally, let

$$\mathcal{E}_{P,\, x_c} = \{x \in \mathbb{R}^n \mid E(x) \le 1\} \subset \mathbb{R}^n. \tag{6}$$

If for all $x \in \mathbb{R}^n$ with trajectories generated according to (3)*, and for all $i \in \{1,\, \cdots,\, N\}$ we have*

$$E(A_i\, x + w_i) \le 1 \;\; \textit{when} \;\; E(A_i\, x + w_i) - E(x) \ge 0 \tag{7}$$

then $\mathcal{E}_{P,\, x_c}$ is positively invariant along trajectories of (2)*. Moreover, $\mathcal{A}_m \subseteq \mathcal{E}_{P,\, x_c}$.*

Proof: Consider the initial condition $x(k) \in \mathcal{E}_{P,\, x_c}$. Here, $E(x(k)) \le 1$. For the case where

$$E(A_i\, x(k) + w_i) - E(x(k)) < 0$$

for a given $i \in \{1, \cdots, N\}$, we have $E(A_i\, x(k) + w_i) < 1$. Therefore, $x(k+1) \in \mathcal{E}_{P,\, x_c}$ for the given value of i.

Next consider the case where, for a given $i \in \{1, \cdots, N\}$, we have

$$E(A_i\, x(k) + w_i) - E(x(k)) \ge 0 \tag{8}$$

By (7), we have $E(A_i\, x(k)+w_i) \leq 1$. Therefore, $x(k+1) \in \mathcal{E}_{P,\,x_c}$ for the given value of i. Since one of the above two cases must hold for all $i \in \{1, \cdots, N\}$, we conclude that $x(k+1) \in \mathcal{E}_{P,\,x_c}$ for all $x(k+1) \in \{A_1\, x(k) + w_1, \cdots, A_N\, x(k) + w_N\}$ and that $\mathcal{E}_{P,\,x_c}$ is positively invariant along trajectories of (2).

The proof that $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$ follows from (7):

$$E(x(k)) > 1 \implies E(A_i\, x(k) + w_i) - E(x(k)) < 0,$$

for all $i \in \{1, \cdots, N\}$. Therefore, for all initial conditions $x(0) \in \mathbb{R}^n$ we have $\lim_{k\to\infty} x(k) \in \mathcal{E}_{P,\,x_c}$, and moreover $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$. □

C. Computing Elliptical Invariant Regions

As shown in Proposition 1, the existence of an $E(x)$ that satisfies (7) guarantees that $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$, where $E(x)$ is defined by (5) and $\mathcal{E}_{P,\,x_c}$ is defined by (6). In what follows, we present an algorithm, encoded as a semidefinite program, which searches for such a mapping $E(x)$. We first present the following theorem.

Theorem 1. *Let $x_c \in \mathbb{R}^n$, and define the mapping $S_i : \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R}^{n+1\times n+1}$:*

$$S_i(P,\,\lambda) \triangleq \begin{bmatrix} S_{1,i} & S_{2,i} \\ S_{2,i}^T & S_{3,i} \end{bmatrix} \tag{9}$$

$$\begin{aligned} S_{1,i} &= (1+\lambda)A_i^T P A_i - \lambda P \\ S_{2,i} &= (1+\lambda)A_i^T P(w_i - x_c) + \lambda P x_c \\ S_{3,i} &= (1+\lambda)w_i^T P(w_i - 2x_c) + x_c^T P x_c - 1 \end{aligned}$$

where $i \in \{1, \cdots N\}$. If there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a positive real $\lambda \in \mathbb{R}^*$ such that $S_i(P,\lambda)$ is negative semidefinite for all $i \in \{1, \cdots, N\}$, then $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$.

Proof: We show this result using certain properties of convex functions. Specifically, we note that by the S-Procedure [15], the condition (7) holds if there exists a $\lambda \in \mathbb{R}^*$ such that

$$1 - E(A_i x + w_i) - \lambda E(A_i x + w_i) + \lambda E(x) > 0$$

for all $x \in \mathbb{R}^n$ and all $i \in \{1, \cdots, N\}$. Formulating this statement as a quadratic inequality in the vector $\begin{bmatrix} x^T & 1 \end{bmatrix}^T$, we have that if

$$\begin{bmatrix} x^T & 1 \end{bmatrix} S_i(P,\,\lambda) \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0$$

for all $i \in \{1, \cdots, N\}$ then $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$, where $S_i(P,\,\lambda)$ is given by (9). Therefore, if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a positive real $\lambda \in \mathbb{R}^*$ such that $S_i(P,\lambda)$ is negative semidefinite for all $i \in \{1, \cdots, N\}$, then $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$. □

Theorem 1 shows that if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$, a vector $x_c \in \mathbb{R}^n$ and a $\lambda \in \mathbb{R}^*$ such that $S_i(P,\,\lambda) \preceq 0$ for all $i \in \{1, \cdots N\}$, then $\mathcal{A}_m \subseteq \mathcal{E}_{P,\,x_c}$. From this result, we present an algorithm for over-approximating the minimal attractor of the switched affine system (2) with a suitable invariant ellipsoid. Since the invariant ellipsoid is the solution to a semidefinite program, the algorithm relies on CVX, a convex optimization toolbox made for use with MATLAB [21], [22].

Algorithm 1 Bounding the Attractor Set of the Switched Affine System (2) with an Invariant Ellipsoid

inputs: $\mathcal{C}'$ from (2). Desired ellipsoidal centroid $x_c \in \mathbb{R}^n$. Free parameter $\lambda \in \mathbb{R}^*$
output: $P \in \mathbb{R}^{n\times n}$, such that $\mathcal{E}_{P,\,x_c}$ from (6) is invariant and over-approximates $\mathcal{A}_m$.

1: **function** FINDINVARIANTSET($\mathcal{C}'$, x_c, λ)
2: **cvx_begin sdp**
3: **variable** $P(n,n)$ **semidefinite**
4: **for** $i = 1$ to N **do**
5: $S_{1,i} := (1+\lambda)A_i^T P A_i - \lambda P$
6: $S_{2,i} := (1+\lambda)A_i^T P(w_i - x_c) + \lambda P x_c$
7: $S_{3,i} := (1+\lambda)w_i^T P(w_i - 2x_c) + x_c^T P x_c - 1$
8: $S_i := \begin{bmatrix} S_{1,i} & S_{2,i} \\ S_{2,i}^T & S_{3,i} \end{bmatrix}$
9: $S_i \leq 0$
10: %% Possibly Insert Objective Function
11: **cvx_end**
12: **if** Program feasible **then**
13: **return** P
14: **else**
15: **return** 'infeasible'
16: **end function**

Algorithm 1 takes as inputs a desired ellipsoid center x_c and a parameter $\lambda \in \mathbb{R}^*$ and returns a positive definite matrix $P \in \mathbb{R}^{n\times n}$, should one exist, which satisfies the constraint (7). Such a feasible solution P will identify $\mathcal{E}_{P,\,x_c}$ as an invariant hyper-ellipsoid in the dimension of the statespace which over-approximates the minimal attractor set of the system (2); this is a result of Theorem 1. In some sense the choice of x_c is arbitrary; if (2) does not diverge, then for all $x_c \in \mathbb{R}^n$ there exists a P and λ which solves the semi-definite program. For this reason, we suggest two methods for selecting a suitable ellipsoid center $x_c \in \mathbb{R}^n$; one may choose to identify the approximate centroid of the attractor set of $\mathcal{A}_m$ through simulation, or instead one may choose to select $x_c \in \mathbb{R}^n$ at the mean of the affine equilibria given in (4). Additionally, note that the solution to the semi-definite program presented in Algorithm 1 is parameterized by λ; a line search can be conducted over this parameter in order to find a feasible outer-approximation of $\mathcal{A}_m$. Since it is in general preferable to compute an ellipsoid which bounds the attractor set as tightly as possible, line 10 in Algorithm 1 can be replaced with an objective function such as

9: **maximize** log_det(P)

where log_det($\cdot$) is a function provided by CVX which computes the natural logarithm of the determinant of the input symmetric matrix and is useful for finding the minimum-

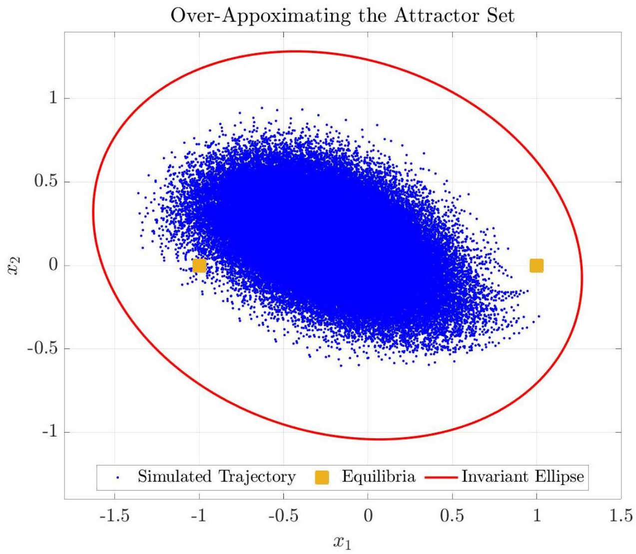


Fig. 1: Over-approximating the minimal attractor set of the system (10). When beginning at an initial position $x(0) = [0,\ 0]^T$, the system (10) can only reach the region shown in blue. An invariant ellipse $\mathcal{E}_{P,\,x_c} \subset \mathbb{R}^2$, shown in red, is calculated using Algorithm (1).

volume ellipsoid.

D. Numerical Example

In this section, we present a sample case and over-approximate the minimal attractor set of a stable switched affine system using Algorithm 1.

Consider the planar shifted rotating system

$$x(k+1) = A_i\, x(k) + w_i, \quad i \in \{1,\, 2\}$$
$$A_1 = .9\begin{bmatrix}\cos(.2) & -\sin(.2)\\ \sin(.2) & \cos(.2)\end{bmatrix}, \quad w_1 = \begin{bmatrix}-0.12\\ 0.19\end{bmatrix} \tag{10}$$
$$A_2 = .9\begin{bmatrix}\cos(.1) & -\sin(.1)\\ \sin(.1) & \cos(.1)\end{bmatrix}, \quad w_2 = \begin{bmatrix}0.10\\ -0.09\end{bmatrix}$$

This example is inspired by the fact that repeating rotation with a nonzero equilibrium forms a fractal-like pattern [23]. Additionally, for this specific choice of system parameters, note that

$$x_{\text{eq},1} = \begin{bmatrix}1\\ 0\end{bmatrix}, \quad x_{\text{eq},2} = \begin{bmatrix}-1\\ 0\end{bmatrix}$$

where $x_{\text{eq},1}$ and $x_{\text{eq},2}$ are given by (4).

We compute an elliptical over-approximation of $\mathcal{A}_m$ of (10) using Algorithm 1. Using inputs $x_c = [-0.18,\ 0.12]^T$ and $\lambda = 9.7$, the semi-definite program is found to be feasible. From the solution $P \in \mathbb{R}^{2\times 2}$, a invariant ellipse is plotted in Figure 1.

III. Time-Varying Switched Systems with Additive Noise

We now return to the initial stochastic setting of (1), restated as follows:

$$x(k+1) = A(k)x(k) + w(k)$$
$$w(k) \sim \mathcal{N}(\mu(k),\, \Sigma(k))$$

where $x(k) \in \mathbb{R}^n$ denotes the system state, $w(k) \in \mathbb{R}^n$ denotes an additive Gaussian white noise term with mean $\mu(k) \in \mathbb{R}^n$ and noise covariance matrix $\Sigma(k) \in \mathbb{R}^{n\times n}$, and where $\mathcal{C} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ denotes a finite set of system parameters, such that $(A(k),\, \mu(k),\, \Sigma(k)) \in \mathcal{C}$ for all $k \in \mathbb{N}$. In this setting, the system state $x(k)$ is a multivariate random variable.

A. Computing a Bound on the State Covariance

In the instance that $\Sigma_i \neq 0$ for $i \in \{1, \cdots N\}$ the system (1) will not have any equilibria. Moreover, at any given time $k \in \mathbb{N}$ the noise term $w(k)$ can be arbitrarily large, and, as such, the infinite-time behavior of the system state $x(k)$ cannot be classified with an invariant region in the state space. For this reason, we instead characterise the system (1) with an invariant ellipsoid which bounds the covariance matrix of the system state. The symbol $\overline{x}(k)$ denotes the mean of the state and $\mathbb{E}[\,\cdot\,]$ denotes the expected value function such that

$$\overline{x}(k) = \mathbb{E}[x(k)] \tag{11}$$

Importantly, at a given time $k \in \mathbb{N}$ there exists an $i \in \{1,\, \cdots,\, N\}$ such that

$$\overline{x}(k+1) = A_i\overline{x}(k) + \mu_i$$

We next define the state and disturbance covariance matrices.

Definition 2 (State and Disturbance Covariance Matrices)**.** We use the symbol $X(k) \in \mathbb{R}^{n\times n}$ to denote the state covariance matrix of the system (1) at a time $k \in \mathbb{N}^*$, which is defined by

$$X(k) \triangleq \mathbb{E}[\,(x(k) - \overline{x}(k))(x(k) - \overline{x}(k))^T\,] \tag{12}$$

As before, we use the symbol $\Sigma_i(k) \in \mathbb{R}^{n\times n}$ to denote the disturbance covariance matrix of the i^{th} mode of the system (1):

$$\Sigma_i \triangleq \mathbb{E}[(w(k) - \mu_i)(w(k) - \mu_i)^T] \tag{13}$$

By virtue of the fact $w(k)$ is a white noise random process, we have that $x(k)$ and $w(k)$ are independent. Therefore

$$\mathbb{E}[\,(x(k) - \overline{x}(k))(w(k) - \mu_i)^T\,] = 0$$

along trajectories of (1). Moreover, the propagation of (12) is governed by a discrete-time Lyapunov recursion:

$$\begin{aligned}X(k+1) &= \mathbb{E}[\,(x(k+1) - \overline{x}(k+1))\cdots\\ &\qquad\cdots(x(k+1) - \overline{x}(k+1))^T]\\ &= A_i\mathbb{E}[\,(x(k) - \overline{x}(k))(x(k) - \overline{x}(k))^T\,]A_i^T + \cdots\\ &\qquad\cdots\mathbb{E}[\,(w(k) - \mu_i)(w(k) - \mu_i)^T\,]\\ &= A_iX(k)A_i^T + \Sigma_i\end{aligned}$$

for some $i \in \{1, \cdots, N\}$, as described in [24]. Taking $\overrightarrow{X} = \text{vec}(X) \in \mathbb{R}^{n^2}$ and $\overrightarrow{\Sigma_i} = \text{vec}(\Sigma_i) \in \mathbb{R}^{n^2}$ to be the vectorizations of the state and disturbance covariance matrices, we then have

$$\overrightarrow{X}(k+1) = \mathcal{A}_i\overrightarrow{X}(k) + \overrightarrow{\Sigma}_i, \quad i \in \{1,\, \cdots,\, N\}, \tag{14}$$

where $i \in \{1, \cdots, N\}$ and $\mathcal{A}_i \triangleq A_i \otimes A_i \in \mathbb{R}^{n^2 \times n^2}$ with $\otimes$ denoting the Kronecker product. In what follows, we refer to the system (14) as the "augmentation" of the initial switched system (1).

Importantly, the augmented system (14) is switched linear affine, as was the case with (2). As such, we can now use Algorithm 1 to compute a hyper-ellipsoidal over-approximation of the attractor set of (14), thus providing a guaranteed bound on the infinite time behavior of the state covariance matrix $X(k)$.

B. Numerical Examples

We consider a planar shifted rotating system, as in (10), now with added Gaussian noise. In this setting, the system (1) has the following system parameters

$$\begin{aligned} A_1 &= 0.9 \begin{bmatrix} \cos(0.2) & -\sin(0.2) \\ \sin(0.2) & \cos(0.2) \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ A_2 &= 0.9 \begin{bmatrix} \cos(0.1) & -\sin(0.1) \\ \sin(0.1) & \cos(0.1) \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{15}$$

To bound the covariance of the state, we first form the switched affine system (14). Note that the mean μ_i of the noise term w_i does not affect the evolution of the state covariance matrix X. Since the matrices given by (12) and (13) are symmetric, the dimension of $\overrightarrow{X}$ can be reduced from four to three. For this example, we reduce the dimension of $\mathcal{A}_i$ and of $\overrightarrow{\Sigma}_i$ accordingly and form the equivalent system

$$\underline{\overrightarrow{X}}(k+1) = \tilde{\mathcal{A}}_i \underline{\overrightarrow{X}}(k) + \underline{\overrightarrow{\Sigma}}_i, \quad i \in \{1, 2\} \tag{16}$$

where, for entries a_{jk} of A_i, $j, k \in \{1, 2\}$

$$\tilde{\mathcal{A}}_i = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{21}a_{11} & a_{21}a_{12} + a_{11}a_{22} & a_{22}a_{12} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}$$

and

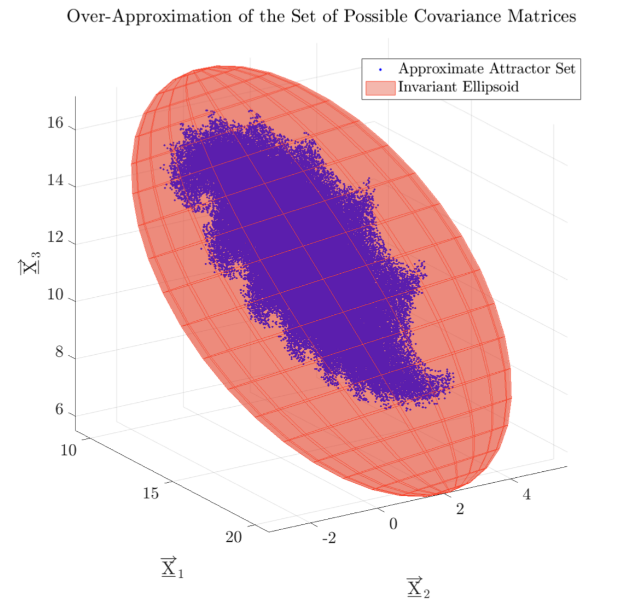
$$\overrightarrow{\Sigma}_i = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \end{bmatrix} \implies \underline{\overrightarrow{\Sigma}}_i = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix}$$

Much is known about how to similarly reduce the order of higher-dimensional systems.

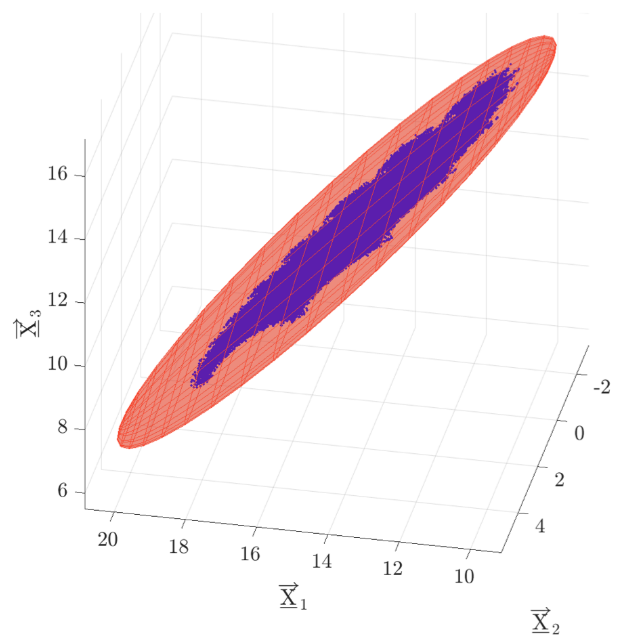
The resulting system (16) is input into Algorithm 1 with the parameters

$$\lambda = 5.21, \; x_c = \begin{bmatrix} 14.98 \\ 1.24 \\ 11.34 \end{bmatrix}$$

The algorithm computes a positive definite matrix P such that $(\underline{\overrightarrow{X}})^T P \underline{\overrightarrow{X}} = 1$ is the three-dimensional invariant ellipsoid which overapproximates the attractor set of the switching system (16). For this example, Algorithm 1 was able to find a minimal P satisfying (9) in 2.6 seconds. A plot of the attractor set and the bounding ellipsoid is shown in Figure 2. As seen in the Figure, the attractor set resembles a fractal structure which seems to live on a plane, and the ellipsoid tightly bounds it. All possible covariance matrices of the state x for the system (1) are contained within the ellipsoid.



(a) First view of ellipsoidal bound and (approximate) minimal attractor of system (14), which features a fractal structure since the state covariance matrix propagates with switched affine dynamics.



(b) Second view. The minimal attractor set is planar, so its ellipsoidal bound is significantly compressed along one dimension.

Fig. 2: Bounding ellipsoid and approximate minimal attractor set of the state covariances of (1) with parameters (15).

The choice of inputs λ and x_c are clearly arbitrary; in this case, a local line search on λ was performed to maximize the function log_det(P), and a coordinate near the centroid of the attractor set was chosen for x_c.

Though the attractor set for the covariances of system (1) with parameters (15) exists on a plane, that is not the

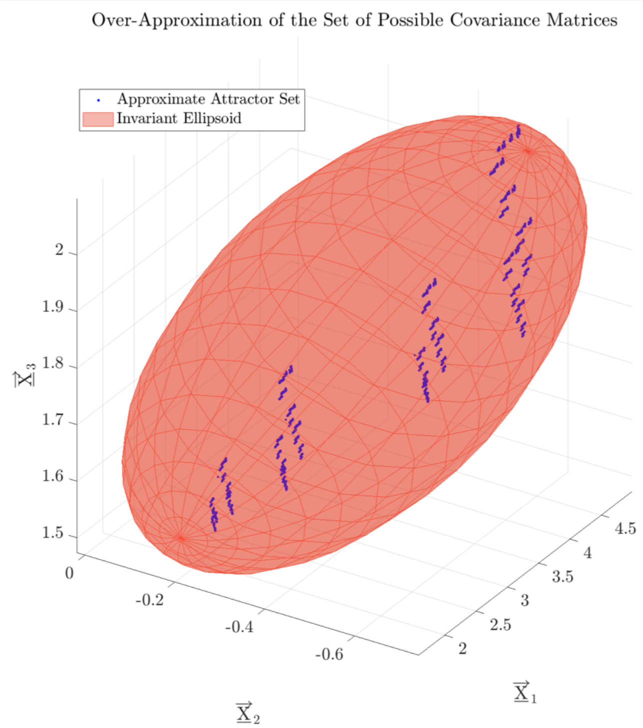


Fig. 3: Bounding ellipsoid and approximate (disconnected) minimal attractor set of the state covariances of (1) with parameters (17).

case in general. In the following example, the attractor set is disconnected, yet an ellipsoidal bound is still computed by Algorithm 1. With the following parameters for system (1),

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.7 & -0.7 \\ 0.2 & 0.7 \end{bmatrix} \quad \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0.6 & -0.3 \\ 0.1 & 0.6 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{17}$$

a bound on the state covariance matrix X is computed by Algorithm 1 and plotted in Figure 3. The free parameters $\lambda = 1.8$ and $x_c = [3.29,\ -0.36,\ 1.79]^T$ were used.

IV. Conclusion

We studied the infinite-time behavior of a time-varying linear system with additive Gaussian noise, formulating an algorithm to compute a bound on the convariance matrix of the system state. To generate the algorithm, a linear affine difference equation which governs the evolution of the state covariance matrix was derived. This is the augmented system, the state of which approaches an attractor set in infinite time. A bound on this attractor set is a bound on the set of possible state covariance matrices for the originial system. The bounding condition was transformed into a matrix inequality such that an invariant ellipsoid can be computed by an SDP solver.

References

[1] D. Liberzon, *Switching in systems and control*. Springer Science & Business Media, 2003.

[2] R. Goebel, R. G. Sanfelice, and A. R. Teel, “Hybrid dynamical systems,” *IEEE Control Systems Magazine*, vol. 29, pp. 28–93, April 2009.

[3] R. W. Brockett, *Hybrid Models for Motion Control Systems*, pp. 29–53. Boston, MA: Birkhäuser Boston, 1993.

[4] V. A. Yakubovich, “The solution of some matrix inequalities encountered in automatic control theory,” *Societ Math Dokl.*, vol. 143, no. 5, pp. 652–656, 1964.

[5] B. R. Barmish, “Necessary and sufficient conditions for quadratic stabilizability of an uncertain system,” *Journal of Optimization Theory and Applications*, vol. 46, pp. 399–408, Aug 1985.

[6] M. Johansson and A. Rantzer, “Computation of piecewise quadratic Lyapunov functions for hybrid systems,” in *1997 European Control Conference (ECC)*, pp. 2005–2010, IEEE, 1997.

[7] D. Mignone, G. Ferrari-Trecate, and M. Morari, “Stability and stabilization of piecewise affine and hybrid systems: An LMI approach,” in *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)*, vol. 1, pp. 504–509, IEEE, 2000.

[8] E. Feron, *Quadratic stabilizability of switched systems via state and output feedback*. Center for Intelligent Control Systems, Massachusetts Institute of Technology, 1996, http://www.feron.org/Eric/quad_stab.pdf.

[9] W. Zhang, A. Abate, J. Hu, and M. P. Vitus, “Exponential stabilization of discrete-time switched linear systems,” *Automatica*, vol. 45, no. 11, pp. 2526–2536, 2009.

[10] D. Chatterjee and D. Liberzon, “On stability of randomly switched nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2390–2394, 2007.

[11] L. Zhang, P. Shi, and M. Basin, “Robust stability and stabilisation of uncertain switched linear discrete time-delay systems,” *IET Control Theory & Applications*, vol. 2, no. 7, pp. 606–614, 2008.

[12] M. Kermani and A. Sakly, “Stability analysis for a class of switched nonlinear time-delay systems,” *Systems Science & Control Engineering: An Open Access Journal*, vol. 2, no. 1, pp. 80–89, 2014.

[13] J. Daafouz, P. Riedinger, and C. Iung, “Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach,” *IEEE Transactions on Automatic Control*, vol. 47, no. 11, pp. 1883–1887, 2002.

[14] X. Liu and X. Zhao, “Stability analysis of discrete-time switched systems: a switched homogeneous Lyapunov function method,” *International Journal of Control*, vol. 89, no. 2, pp. 297–305, 2016.

[15] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*, vol. 15. Siam, 1994.

[16] M. Abate, C. Klett, S. Coogan, and E. Feron, “Lyapunov differential equation hierarchy and polynomial Lyapunov functions for switched linear systems,” *arXiv preprint arXiv:1906.04810v3*, 2020.

[17] J. Milnor, “On the concept of attractor,” in *The theory of chaotic attractors*, pp. 243–264, Springer, 1985.

[18] E. Feron and C. Olivier, “Targets, sensors and infinite-horizon tracking optimality,” in *29th IEEE Conference on Decision and Control*, pp. 2291–2292, IEEE, 1990.

[19] K. Okamoto, M. Goldshtein, and P. Tsiotras, “Optimal covariance control for stochastic systems under chance constraints,” *IEEE Control Systems Letters*, vol. 2, no. 2, pp. 266–271, 2018.

[20] J. P. Hespanha, D. Liberzon, D. Angeli, and E. D. Sontag, “Nonlinear norm-observability notions and stability of switched systems,” *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 154–168, 2005.

[21] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 2.1.” `http://cvxr.com/cvx`, Mar. 2014.

[22] M. Grant and S. Boyd, “Graph implementations for nonsmooth convex programs,” in *Recent Advances in Learning and Control* (V. Blondel, S. Boyd, and H. Kimura, eds.), Lecture Notes in Control and Information Sciences, pp. 95–110, Springer-Verlag Limited, 2008. `http://stanford.edu/~boyd/graph_dcp.html`.

[23] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Fractals for the classroom: part one introduction to fractals and chaos*. Springer Science & Business Media, 2013.

[24] D. Simon, *Optimal state estimation: Kalman, H_∞, and nonlinear approaches*. John Wiley & Sons, 2006.