A Numerical Method to Compute Stability Margins of Switching Linear Systems

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Abstract— **Stability margins for linear time-varying (LTV) and switched-linear systems are traditionally computed via quadratic Lyapunov functions, and these functions certify the stability of the system under study. In this work, we show how the more general class of homogeneous polynomial Lyapunov functions is used to compute stability margins with reduced conservatism, and we show how these Lyapunov functions aid in the search for periodic trajectories for marginally stable LTV systems. Our work is premised on the recent observation that the search for a homogeneous polynomial Lyapunov function for some LTV systems is easily encoded as the search for a quadratic Lyapunov function for a related LTV system, and our main contribution is an intuitive algorithm for generating upper and lower bounds on the system's stability margin. We show also how the worst-case switching scheme—which draws an LTV system *closest* to a periodic orbit—is generated. Three numerical examples are provided to aid the reader and demonstrate the contributions of the work.**

I. Introduction

Switched-linear and linear time-varying (LTV) systems are two important and widely studied classes of systems [1]. They are useful abstractions for studying systems with uncertain parameters, and they can be used to represent mode-switched systems when the switching scheme may be unknown. In all cases, computing stability margins is an important exercise, with practical benefits for safety critical systems [2]. In the following, we study both switched-linear and LTV systems in an equivalent manner and propose methods to compute upper and lower bounds on a system's stability margin.

We study the stability and robustness of LTV systems whose dynamics are influenced by an uncertain or time-varying system parameter. We characterize the system's robustness with respect to this parameter by bounding from above and below the parameter value for which a diverging trajectory can be produced. As will be shown, these bounds hint at the existence of a specific parameter value that can induce periodic orbits for LTV systems but cannot cause diverging trajectories.

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Stability margins are generally impossible to compute exactly for all but linear time-invariant systems, though much work has been done on estimating the margin through the computation of upper and lower bounds [3][4]. For time-varying systems, while quadratic stability has been studied in depth since the late 1940s by Lur'e, Yakubovitch, and others [5], exact approaches to the computation of stability margins are more recent and include [6] and [7]. However, most computationally tractable analyses are generally conservative and typically give results which may significantly under- or over-approximate the true stability margin.

In order to gain an understanding of the conservatism of these types of bounding methods, we compute a less-conservative bound by searching for a trajectory which becomes unstable when the system is allowed to operate near its stability margin. Such a trajectory is referred to as a *counter-example* and is a powerful tool for numerically approximating the true stability margin. Generating a counter-example can be challenging. In many cases, simulations will show that a time-varying system operating near or even beyond its stability margin is always stable. Recent work uses sum-of-squares programming to investigate computation of worst-case switching sequences for discrete time systems [8][9].

In the case where the system matrix remains Hurwitz for all time, the examples studied indicate that instability occurs through a switching sequence which leads the system through a Hopf bifurcation as the uncertain parameter is adjusted. A theoretical result which confirms this observation is not achieved, but the observation nonetheless provides the intuition behind the algorithm for computing a counter-example and therefore an upper bound on the stability margin. In classical linear systems analysis, a system which depends on a single parameter moves from stable to unstable when, as the parameter is adjusted, the poles on the root locus cross through the origin or through the imaginary axis. In the latter case, instability is achieved via a Hopf bifurcation. For the LTV system currently under study, the system matrix can remain Hurwitz so that all poles stay in the left-half plane for all time, but the time-dependent variation of the uncertain parameter can induce a limit cycle.

Our approach to bounding stability margins is aided by constructing quadratic Lyapunov functions for a lifted system [10]. In [11], the search for homogeneous polynomial Lyapunov functions for LTV systems is recast as a search for quadratic Lyapunov functions for a related hierarchy of time-varying Lyapunov differential equations; thus, this is an elegant and equivalent alternative to sum-of-squares programming techniques for computing Lyapunov functions [12]. Intuitive procedures following from these observations have been shown to provide higher-fidelity performance analyses [13] [14]. The Lyapunov functions returned by these analyses also allow us to improve stability margin analysis by providing valuable information about how to find counter-examples or worst-case trajectories.

II. Notation and Preliminaries

We denote by $S^n_{++} \subset \mathbb{R}^{n\times n}$ the set of symmetric positive definite $n \times n$ matrices. We denote by I_n the $n \times n$ identity matrix, and we denote by $0_n \in \mathbb{R}^n$ the zero vector in $\mathbb{R}^n$. Given $M \in \mathbb{R}^{n\times m}$ and integer $i \geq 1$, we denote by $\otimes^i M \in \mathbb{R}^{n^i\times m^i}$ the i^{th}-Kronecker power of M, as defined recursively by

$$\begin{aligned} &\otimes^1 M := M \\ &\otimes^i M := M \otimes (\otimes^{i-1} M) \qquad i \geq 2. \end{aligned} \tag{1}$$

We study, in the following, switched linear systems, as in

$$\dot{x} = A(t)x, \tag{2}$$

with state $x \in \mathbb{R}^n$ and where $A(t) \in \mathcal{S} \subseteq \mathbb{R}^{n\times n}$ evolves for all time inside a finite set of switched modes

$$\mathcal{S} = \{A_1, \cdots, A_k\}. \tag{3}$$

Definition 1 (Stability)**.** The system (2) is *stable* if for all initial conditions $x(0) \in \mathbb{R}^n$, we have that $\lim_{t\to\infty} x(t) = 0_n$ for all $A(t)$ satisfying (3). ■

The system (2) is *unstable* when (2) is not stable.

Definition 2 (Quadratic Stability)**.** For $P \in S^n_{++}$, the function $V(x) = x^T P x$ is a quadratic Lyapunov function for (2) if

$$A^T P + PA \prec 0 \quad \text{for all} \quad A \in \mathcal{S}. \tag{4}$$

■

When (2) is *linear time-invariant*—equivalently, when $k = 1$—the system (2) is stable if and only if there exists a quadratic Lyapunov function for (2). This is, however, not true in the general setting of (2); indeed, there exists stable switched systems for which no quadratic Lyapunov function exists [1]. Nonetheless, the existence of such a quadratic Lyapunov function for (2) guarantees the stability of the system.

It was recently shown in [11] how a related hierarchy of switched linear systems is used to assess the stability of (2). Consider the following infinite hierarchy of switched linear systems:

$$\begin{aligned} &H_1 : \left\{ \begin{array}{rcl} \dot{\xi}_1 &=& \mathcal{A}_1(t)\xi_1 \\ \mathcal{A}_1(t) &\in& \mathcal{S}_1 \\ S_1 &=& \{\mathcal{A}^1_1, \cdots, \mathcal{A}^1_k\} \\ \mathcal{A}^1_j &=& A_j \end{array} \right. \\ &H_i : \left\{ \begin{array}{rcl} \dot{\xi}_i &=& \mathcal{A}_i(t)\xi_i \\ \mathcal{A}_i(t) &\in& \mathcal{S}_i \\ \mathcal{S}_i &=& \{\mathcal{A}^i_1, \cdots, \mathcal{A}^i_k\} \\ \mathcal{A}^i_j &=& I_n \otimes \mathcal{A}^{i-1}_j + A_j \otimes I_{n^{i-1}} \end{array} \right. \end{aligned} \tag{5}$$

where $\xi_i \in \mathbb{R}^{n^i}$ is the state of the i^{th}-level system H_i and $i \geq 1$. Each system H_i in the hierarchy is switched linear, as in (2).

The hierarchy (5) is best understood by looking at the first and second level systems H_1 and H_2. The system H_1 is equivalent to the switched linear system (2), and the system H_2 is nothing but the vectorized version of the Lyapunov differential equation

$$\dot{X}(t) = A(t)X(t) + X(t)A(t)^T \tag{6}$$

where $A(t)$ maintains its definition from (2) and where the state of (6) is a matrix $X \in \mathbb{R}^{n\times n}$. Moreover, it is shown in [11] that if $x(t)$ is a solution to (2), then $\xi_i(t) = (\otimes^i x(t))$ is a solution to H_i. It follows, therefore, that if the i^{th}-level H_i is stable, then the system (2) is stable as well.

Proposition 1 ([11])**.** *If the i^{th}-level system H_i is quadratically stable, i.e. if there exists a $P \in S^{n^i}_{++}$ such that*

$$\mathcal{A}^T P + P\mathcal{A} \prec 0 \quad \textit{for all} \quad \mathcal{A} \in \mathcal{S}_i, \tag{7}$$

then the system (2) *is stable and*

$$V(x) = (\otimes^i x)^T P (\otimes^i x) \tag{8}$$

is a homogeneous polynomial Lyapunov function for (2) *of order* $2i$.

Note that the system (2) can be stable even when H_i is not quadratically stable for a given $i \geq 1$. Nonetheless, the stability of (2) implies that there must exist an $i \geq 1$ such that H_i is quadratically stable [11].

III. Problem Statement

In the following, we consider linear time-varying (LTV) systems

$$\begin{aligned} \dot{x} &= A(t)x \\ A(t) &= A + \Delta(t)A_0 \end{aligned} \tag{9}$$

for $A, A_0 \in \mathbb{R}^{n\times n}$ and time-varying scalar $\Delta(t) \geq 0$. We refer to A as the nominal dynamics, and the term $\Delta(t)A_0$ describes a time-varying perturbation. We assume that A is Hurwitz so that (9) is stable when $\Delta(t) \equiv 0$.

Remark 1. The switched linear system (2) is a generalization of (9); in particular, if $\mathcal{S}$ in (3) is taken as

$$\mathcal{S} = \{A, A + \delta A_0\}, \tag{10}$$

then the switched system (2) is stable if and only if the linear time varying system (9) is stable for all $\Delta(t)$ that satisfy $\Delta(t) \in [0, \delta]$ for all $t \geq 0$. ■

Definition 3. Given $A_0, A \in \mathbb{R}^{n\times n}$ and $\delta \in \mathbb{R}$, the system (9) is stable with respect to δ if (9) is stable for all $\Delta(t)$ such that $\Delta(t) \in [0, \delta]$ for all $t \geq 0$. The *stability margin* for (9) is the unique $\hat{\delta} \geq 0$ such that (9) is stable with respect to all $\delta \in [0, \hat{\delta})$ and such that (9) is not stable with respect to $\hat{\delta}$. ■

The system (9) is guaranteed to be stable with respect to all $\delta \in [0, \hat{\delta})$, and for this reason, we begin by studying methods for under-approximating the stability margin for (9).

Problem 1. Under-approximate the stability margin $\hat{\delta}$ for (9) by finding the largest possible δ for which a Lyapunov function can be computed which certifies that (9) is stable with respsect to δ. We denote this lower bound on $\hat{\delta}$ as $\underline{\delta}$. Numerous methods exist for computing $\underline{\delta}$ [5], but these methods generally provide conservative estimates of the stability margin $\hat{\delta}$. ∎

Next, consider the case that $A + \delta A_0$ remains Hurwitz and assume that, for $\delta = \hat{\delta} + \varepsilon$, there exists an $\varepsilon > 0$ and a switching function $\Delta(t)$ which produces a diverging trajectory. Based on the examples studied, we observe that, when $\varepsilon = 0$, there exists a $\Delta(t)$ that can induce a periodic trajectory but not a diverging trajectory. Furthermore, for any $\varepsilon > 0$ there exists a $\Delta(t)$ which can produce a diverging trajectory. This observation has not been proven; however, it inspires the development of a useful algorithm for bounding the stability margin from above and finding a periodic trajectory. Even if the observation cannot be proven, an upper bound on the stability margin found by producing a limit cycle can be considered a useful estimate in many cases.

Definition 4. A switching signal $\Delta(t) \in [0,\, \delta]$ which can cause a trajectory of (9) to become periodic when $\delta = \hat{\delta}$ is called a *worst-case switching function* and is denoted as $\Delta_w(t;\delta)$. The system matrix $A(t)$ produced by the worst-case switching function is denoted by $A_w(t;\delta) = A + \Delta_w(t;\delta)A_0$. ∎

Remark 2. It is not proven that $\Delta_w(t;\delta)$ always exists, but in practice it has been approximated for every example LTV system studied. It is also not necessarily unique. Different switching functions can be found by studying (9) with different initial conditions, but non-uniqueness is not important since any $\Delta_w(t;\delta)$ will produce a periodic trajectory when $\delta = \hat{\delta}$, and it is the approximation of this $\hat{\delta}$ which is of primary interest. ∎

Even for $\delta < \hat{\delta}$, a worst-case switching function $\Delta_w(t;\delta)$ is very useful for system analysis as it can be used to slow a trajectory's convergence to the origin. See Example 3 in Section VII.

While $\Delta_w(t;\delta)$ cannot be computed exactly, it can be closely approximated using a Lyapunov function for (9), and a higher-order Lyapunov function will produce a better approximation. We observe that the transition from stability to instability occurs through a switching sequence which results in a periodic orbit about the surface of a level set of a Lyapunov function.

Problem 2. Given A, A_0, δ and a Lyapunov function parameterized by P from the solution to Problem 1, approximate $\Delta_w(t;\delta)$ and $A_w(t;\delta)$. These approximations are denoted as $\Delta_w(t;\delta,P)$ and $A_w(t;\delta,P)$. It is convenient to include the order of the Lyapunov function used to generate the switching function as a subscript, as in $\Delta_w(t;\delta,P_{2i})$. ∎

Once an approximate worst-case switching function is found, the smallest value of δ for which $\Delta_w(t;\delta,P)$ produces a periodic trajectory can be computed.

Problem 3. Find the smallest value of δ such that the system (9) with $A_w(t;\delta,P)$ produces a periodic trajectory. In this case, δ approximates the lowest upper bound on the stability margin $\hat{\delta}$. This upper bound is denoted as $\overline{\delta}$. ∎

IV. Under-Approximating the Stability Margin

We begin by addressing Problem 1. The system (9) is stable if and only if there exists a homogeneous sum-of-squares polynomial Lyapunov function certifying its stability [15] [16]. Thus, we present an iterative approach for under-approximating $\hat{\delta}$, and this approach relies on the hierarchy of switched systems (5).

Proposition 2. *For a given $i \geq 1$ and $\delta \geq 0$, assume that there exists a $P \in S_{++}^{n^i}$ satisfying the constraint* (7) *for $\mathcal{S} = \{A,\, A + \delta A_0\}$ such that* (9) *is stable with respect to δ. Define by*

$$\delta_P := \underset{\delta^* \geq \delta}{\arg\max}\, \delta^* \tag{11}$$

$$\text{s.t.} \qquad (\mathcal{A} + \delta^* \mathcal{A}_0)^T P + P(\mathcal{A} + \delta^* \mathcal{A}_0) \preceq 0 \tag{12}$$

where $\mathcal{A}, \mathcal{A}_0 \in \mathcal{S}_i$ as in (5)*. Then $\underline{\delta} = \delta_P$ solves Problem 1 and* (8) *is a Lyapunov function certifying stability of* (9).

The proof follows directly from Proposition 1 and the discussion provided in Remark 1. By iterating over i, conservatism is reduced; the largest value of i for which the optimization problem in Proposition 2 is computationally tractable results in the best possible estimate for $\underline{\delta}$.

V. Computing the Worst-Case Switching Function

The Lyapunov function from Proposition 2 proves asymptotic stability of (9) with $\Delta(t) \in [0,\, \underline{\delta}]$. We should be able to identify $\hat{\delta}$ with an iterative procedure which uses Proposition 2 since a homogeneous polynomial Lyapunov function is a necessary condition for the stability of (9). However, computing a Lyapunov function of sufficiently high order via semidefinite programming techniques may become computationally intractable for higher dimensional systems or as $\delta \to \hat{\delta}$. Therefore, we lay the groundwork for a numerical procedure to produce an upper bound for $\hat{\delta}$ in this section.

The Lyapunov function from Proposition 2 includes valuable information about how to shape $\Delta(t)$ in order to produce a periodic or diverging trajectory. Specifically, choosing a function $\Delta(t) \in [0,\, \delta]$ which maximizes the time derivative of the Lyapunov function will push the system as close as possible to instability for the given δ, especially as the order of the Lyapunov function is increased. This intuition is formalized in Proposition 3, which we use to approximate the worst-case switching function $\Delta_w(t;\delta)$.

Proposition 3. *Let* (8) *be a Lyapunov function, parameterized by P, for the system* (9)*. Then the function $\Delta_w(t;\delta,P)$ which maximizes $\dot{V}(x(t))$ is given by*

$$\Delta_w(t;\delta,P) = \begin{cases} 0, & \textit{if } \mathcal{I}(t) < 0 \\ \delta, & \textit{otherwise} \end{cases} \tag{13}$$

where the indicator function $\mathcal{I}(t)$ is given by

$$\mathcal{I}(t) = (\otimes^i x(t))^T (\mathcal{A}_0^T P + P\mathcal{A}_0)(\otimes^i x(t)) \tag{14}$$

for $\mathcal{A}_0 = \mathcal{A}_0^i$.

Proof. The optimization problem

$$\begin{gathered} \Delta_w(t;\delta,P) = \arg\max_{\Delta} \dot{V}(x(t)) \\ \text{s.t. } \Delta \geq 0 \text{ and } \Delta \leq \delta \end{gathered} \tag{15}$$

with

$$\begin{aligned} \dot{V}(x(t)) = (\otimes^i x(t))^T ((\mathcal{A} + \Delta\mathcal{A}_0)^T P \\ + P(\mathcal{A} + \Delta\mathcal{A}_0))(\otimes^i x(t)) \end{aligned}$$

for $\mathcal{A} = \mathcal{A}^i$ and $\mathcal{A}_0 = \mathcal{A}_0^i$ is solved by (13). □

The optimization problem (15) shows that the $\Delta(t)$ selected is the value in the interval $[0,\delta]$ which maximizes $\dot{V}(x(t))$ at each $t \geq 0$. The state-dependence of $\mathcal{I}(t)$ suggests that $\Delta_w(t;\delta,P)$ is not unique and generally differs for trajectories beginning at different initial conditions x_0. Therefore, Algorithm 1 uses a simulation given an initial condition x_0 and a parameter δ and numerically solves (14) in order to produce (13). Moreover, the system (9) need not be stable with respect to δ in order to produce (13).

Remark 3. Since $\Delta_w(t;\delta,P)$ is a piecewise-constant function, it is most easily utilized in a numerical procedure when expressed as a pair of finite sets $T = \{t_0,\ldots,t_f\}$ and $\Sigma = \{\sigma_0,\ldots,\sigma_{f-1}\}$ such that

$$\Delta_w(t;\delta,P) = \begin{cases} \sigma_0, & \text{for } t_0 \leq t < t_1 \\ & \ldots \\ \sigma_{f-1}, & \text{for } t_{f-1} \leq t < t_f \end{cases} \tag{16}$$

■

The worst-case switching function (13) contains all of the information needed to compute an upper bound on the stability margin of (9).

VI. Bounding the Stability Margin From Above and Producing a Periodic Trajectory

We now seek to approximate the lowest upper bound on the stability margin of (9) with an iterative procedure which, starting with $\delta = \underline{\delta}$, increments δ until $A_w(t;\delta,P)$ can be shown to produce a periodic trajectory. Such a trajectory with an initial condition x_0 and time horizon t_f can become periodic, but it is not necessarily periodic from x_0. Therefore, the switching sequence described by T and Σ in (16) is searched to see if there exist indices $k \geq j \geq 0$ that allow us to construct a discrete transition matrix A_d with an eigenvalue of magnitude 1.

Algorithm 1 Solve Problem 2

input : A, A_0 from (9), desired margin δ.
P from Proposition 2.
Simulation parameters: initial condition x_0 and time horizon t_f.
output: T, Σ from (16) to describe $\Delta_w(t;\delta,P)$.

function FINDSWITCHINGSEQUENCE(inputs)
  initialize:
    $\mathcal{A}^i$, $\mathcal{A}_0^i \leftarrow$ from (5)
    $t_0 \leftarrow 0$
    $\sigma_0 \leftarrow$ from (13)
    $T \leftarrow \{t_0\}$
    $\Sigma \leftarrow \{\sigma_0\}$
    $k \leftarrow 1$
  while $t_{k-1} < t_f$ **do**
    Numerically solve:
      $t_k \leftarrow t_{k-1} + \arg\min_{t>0} \quad \mathcal{I}(t - t_{k-1}) = 0$
    $T \leftarrow \text{append}(t_k)$
    if $\Sigma(k-1) == 0$ **then**
      $\Sigma(k) \leftarrow \delta$
    else
      $\Sigma(k) \leftarrow 0$
    $k = k + 1$
  return T, Σ
end function

Proposition 4. *A switching function described by* (16) *with sets T and Σ produces a limit cycle for* (9) *if there exists some $k \geq j \geq 0$ such that the discrete transition matrix*

$$\begin{aligned} &A_d = \\ &\quad e^{(A+\Sigma(k-1)A_0)(t_k - t_{k-1})} \cdots e^{(A+\Sigma(j)A_0)(t_{j+1}-t_j)} \end{aligned} \tag{17}$$

has an eigenvalue of magnitude equal to 1.

Algorithm 2 seeks to find the parameter $\overline{\delta}$ which induces a limit cycle. In doing so, it calls Algorithm 1 to compute $\Delta_w(t;\delta,P)$ and searches the switching sequence for a discrete transition matrix A_d that has an eigenvalue of magnitude 1. Such an A_d certifies that $\Delta_w(t;\overline{\delta},P)$ produces a periodic trajectory. Note that a higher-order Lyapunov function from Proposition 2 will result in Algorithm 2 returning a smaller value of $\overline{\delta}$, thereby reducing conservatism of the analysis.

VII. Numerical Examples and Applications

The practical importance of the techniques presented in this article is highlighted in the following examples.

Example 1. [Second Order System] We first study the system (9) with $n = 2$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}, \; A_0 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{18}$$

Algorithm 2 Solve Problem 3

input : A, A_0 from (9).
$\underline{\delta}$, P from Proposition 2.
Simulation parameters: initial condition x_0 and time horizon t_f.

output: $\overline{\delta}$, an upper bound on $\hat{\delta}$.
T, Σ from (16) which produces a periodic trajectory.
A_d, discrete transition matrix certifying marginal stability.

function FINDPERIODICTRAJECTORY(inputs)
  initialize:
    $\delta \leftarrow \underline{\delta}$
  while No Periodic Trajectory Found **do**
    if $A + \delta A_0$ not Hurwitz **then**
      % Trivial Switching Sequence Found
      $A_d = e^{(A+\delta A_0)(t_f)}$
      return $\overline{\delta} = \delta$, A_d, $\Sigma = \{\delta\}$, $T = \{0, t_f\}$
    T, Σ = FINDSWITCHINGSEQUENCE(inputs)
    Search T, Σ for $k \geq j \geq 0$ that satisfies (17).
    if A_d from (17) found **then**
      return $\overline{\delta} = \delta$, A_d, $T = T(j:k) - T(j)$, $\Sigma = \Sigma(j:k-1)$
    $\delta \leftarrow$ increment
end function

This system resembles a spring-mass damper with uncertainty in the spring constant. A lower bound on the stability margin is $\underline{\delta} = 2.15$, which was computed using a 14$^{\text{th}}$ order polynomial Lyapunov function. By increasing i to compute a 28$^{\text{th}}$ order Lyapunov function, the algorithm computes $\underline{\delta} = 2.16$. Beyond that, only slight increases in $\underline{\delta}$ can be obtained for significantly higher-order Lyapunov functions, and computation quickly becomes intractable.

Using the 14$^{\text{th}}$ order Lyapunov function, Algorithm 2 provides an upper bound $\overline{\delta}$ on the system's stability margin and provides the subset of the switching sequence from Algorithm 1 which certifies periodicity of the system (or near periodicity—the numerical method is approximate) with $A_w(t; \overline{\delta}, P)$. The value $\overline{\delta} = 2.21$ was computed, and two switching sequences were returned: one using simulation data from $x_0 = [1\ 1]^T$ and another with $x_0 = [-0.2\ 0.8]^T$, both with a time horizon $t_f = 20$. The difference $\overline{\delta} - \underline{\delta} = 0.06$ illustrates how the approximation of a stability margin computed using a common Lyapunov function as in Proposition 2 is conservative.

Switching Sequence 1 using $x_0 = [1\ 1]^T$	
T_1	{0, 0.943, 2.374, 3.317, 4.747}
Σ_1	{2.21, 0, 2.21, 0}

Switching Sequence 2 using $x_0 = [-0.2\ 0.8]^T$	
T_2	{0, 0.251, 1.681, 2.624, 4.055}
Σ_2	{2.21, 0, 2.21, 0}

Figure 1 shows two trajectories from $x_0 = [-0.2\ 0.8]^T$, one using $A_w(t; \overline{\delta}, P_{14})$ described by (T_1, Σ_1) and the other using $A_w(t; \overline{\delta}, P_{14})$ described by (T_2, Σ_2). Both switching sequences produce a limit cycle, but each limit cycle lies approximately on a different level set of the Lyapunov function. This demonstrates that a function $\Delta_w(t; \delta, P)$ is not unique but that any such function can produce a periodic trajectory for the same value of δ.

■

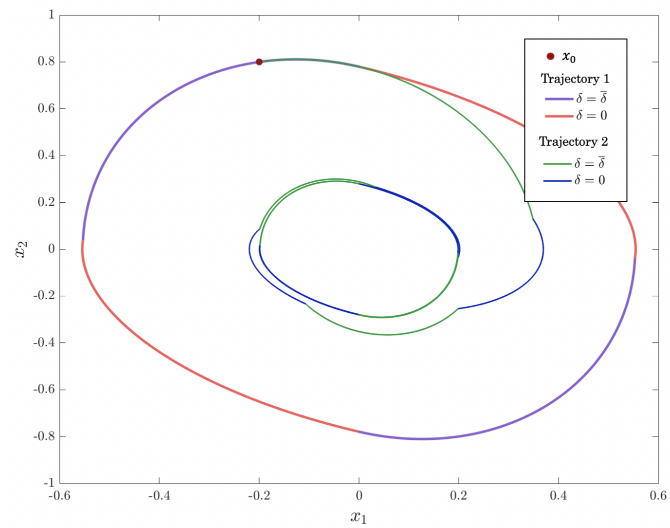

Fig. 1: Example 1. Plot of two trajectories beginning at x_0 but with two different switching functions $\Delta_w(t; \overline{\delta} = 2.21, P_{14})$. Both result in periodic trajectories.

Example 2. [Fourth Order System] For higher-order systems, limit cycles cannot be visualized on the phase plane; instead we search for periodic orbits via analysis of the indicator function (14) and Algorithm 2. To demonstrate this, we study (9) with 4$^{\text{th}}$ order dynamics inspired by the linearized, non-dimensional lateral dynamics of a fixed-wing aircraft with the following parameters [17]:

$$A = \begin{bmatrix} -3.088 & 0 & -1425.042 & 4.5956 \\ -18.906 & -166.878 & 29.223 & 0 \\ 6.762 & 4.445 & -19.389 & 0 \\ 0 & 1428.6 & 0 & 0 \end{bmatrix},$$
$$A_0 = \begin{bmatrix} -1 & 0 & -10 & 10 \\ -10 & -10 & 10 & 0 \\ 10 & 10 & -10 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix}. \tag{19}$$

The best possible lower bound on the stability margin was found to be $\underline{\delta} = 0.24$ for a 6th-order polynomial Lyapunov function. It may be possible to find a larger $\underline{\delta}$, but increasing i to attain a higher-order Lyapunov function would make computation intractable without additionally employing an algorithm to reduce the redundancies produced by the use of the Kronecker product.

To get a better picture of system robustness, we run Algorithm 2 using the 6th-order Lyapunov function and compute $\overline{\delta} = 0.27$. The switching sequence $\Delta(t;\overline{\delta},P_6)$ which produces a periodic trajectory is returned, described by (T,Σ):

Switching Sequence using $x_0 = [1\ 1\ 1\ 1]^T$	
T_1	{0, 0.027, 0.060}
Σ_1	{0.27, 0}

■

Of practical interest to an analysis of uncertain systems are worst-case impulse responses. In [13], the authors use the system hierarchy (5) to easily compute exponential bounds on an impulse response. The methods in this paper can be used to find a worst-case impulse response which approaches these bounds. We add a control input and define an output for the system (9) and study

$$\begin{aligned}\dot{x} &= A(t)x + Bu\\ y &= Cx\\ A(t) &= A + \Delta(t)A_0\end{aligned} \tag{20}$$

The impulse response $h(t)$ of (20) is described parametrically by a solution $\varphi(t)$ to

$$\begin{aligned}\dot{\varphi}(t) &= A(t)\varphi(t)\\ h(t) &= C\varphi(t)\\ \varphi(0) &= B\end{aligned} \tag{21}$$

Since the impulse response of (20) is closely related to the unforced response, the algorithms presented above can be used to compute various worst-case and marginally stable trajectories in this scenario as well.

Example 3 (Worst-Case Impulse Response)**.** Consider the system studied in Example 1 with the additional parameters $B = [0\ 1]^T$ and $C = [1\ 0]$. We use Theorem 4 in [13], with parameter $\alpha = 0.11$, to produce an exponential bound on the impulse response. Using Algorithm 1, a worst-case switching function $\Delta_w(t;\delta = 1, P_{14})$ is computed for (21). Figure 2 shows how this switching function produces an impulse response which approaches the bound much more closely than the system with no switching.

■

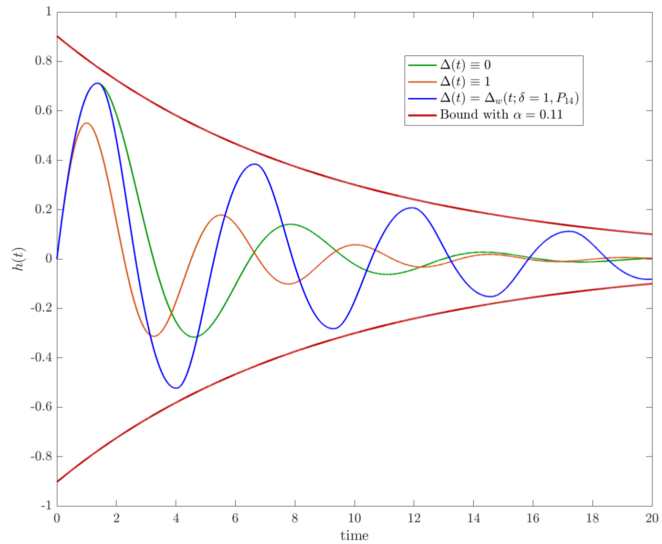

Fig. 2: Example 3. Impulse response plots showing how using the worst-case switching function can cause an impulse response to approach an exponential bound.

VIII. Conclusion

In this work, we show how the more general class of homogeneous polynomial Lyapunov functions is used to compute stability margins with reduced conservatism, and we show how these Lyapunov functions aid in the search for periodic or worst-case trajectories for LTV systems. The main contribution is an intuitive algorithm for generating an upper bound on the system's stability margin and for computing the worst-case switching scheme.

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