A Hierarchy of Quadratic Lyapunov Functions for Linear Time-Varying and Related Systems

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A method for constructing homogeneous polynomial Lyapunov functions is presented for linear time-varying or switched-linear systems and the class of nonlinear systems that can be represented as such. The method uses a simple recursion based on the Kronecker product to generate a hierarchy of related dynamical systems, whose first element is the system under study and the second element is the well-known Lyapunov differential equation. It is then proven that a quadratic Lyapunov function for the system at one level in the hierarchy, which can be found via semidefinite programming, is a homogeneous polynomial Lyapunov function for the system at the base level in the hierarchy. Searching for Lyapunov functions of the foregoing kind is equivalent to searching for homogeneous polynomial Lyapunov functions via the formulation of sum-of-squares programs. The quadratic perspective presented in this paper enables the easy development of procedures to compute bounds on pointwise-in-time system metrics, such as peak norms, system stability margins, and many other performance measures. The applications of the theory to analyzing an aircraft model, on the one hand, and an experimental aerospace vehicle, on the other hand, are presented. The theory can be comprehended with a first course on state-space control systems and an elementary knowledge of convex programming.

I. Introduction

Linear time- and parameter-varying representations of dynamical systems are prevalent in aerospace control applications. For example, they form the basis for the development of the ubiquitous gain-scheduling methods used in flight control systems, including the control of aircraft engines. They are also used to capture the linearized dynamics of a nonlinear dynamical system in the vicinity of one of its trajectories [1, p 59] or to analyze its transverse

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dynamics [2, 3]. Last, linear and parameter-varying systems can be used to conservatively capture the dynamics of nonlinear systems when the nonlinearity is sector-bounded [4].

Influenced by theorems stating the equivalence between the stability of a linear system and the existence of a quadratic Lyapunov function that proves it [5, p277], linear time- and parameter-varying systems are often associated with the computation of quadratic Lyapunov functions to prove their stability or provide valuable bounds on their performance. Early analytical works on systems that exhibit sector-bounded nonlinearities and the identification of quadratic Lyapunov functions that prove their stability include, for example, those of Yakubovitch, who established the celebrated lemma linking the existence of a quadratic Lyapunov function proving the stability of such systems with testing specific properties of a related transfer function in the frequency domain [6]. Since then, the formulation of aerospace control systems as time-varying linear systems whose stability is proved by means of quadratic Lyapunov functions has been spreading to many applications, helped by the parallel development of efficient computational procedures, notably based on convex optimization [4]. Close to the authors, a gain-scheduled control system for a gas-turbine engine leverages such a construction in [5] to prove closed-loop quadratic stability. In [6], a novel triple-integrator maneuver-regulation controller is used to produce a reduced-gravity environment on board an experimental quadrotor vehicle. A parameter-varying representation of the system's transverse dynamics is used to guarantee that the maneuver is stable via the identification of a suitable quadratic Lyapunov function. These are, by far, not the first or only aerospace applications of time- or parameter-varying systems whose stability and performance is supported by quadratic Lyapunov functions: The research literature contains many other aerospace applications of quadratic Lyapunov functions for control system design and analysis; see for example [7–11]. Among them, pioneering works [12] go as far as using a linear parameter-varying representation of aerospace systems to simultaneously design control laws and the quadratic Lyapunov functions that prove their closed-loop stability and their performance using convex optimization techniques.

The popularity of quadratic Lyapunov functions in system stability and performance analysis should not hide, however, the fact that they form a conservative analysis framework: Given a time-varying linear system, the non-existence of a quadratic Lyapunov function does not imply that the system is not stable. This conservatism has led researchers to look for less conservative ways to establish the stability of linear, time-varying and related systems. Among them, some of the more attractive methods have focused on the computer-assisted computation of higher-order polynomial Lyapunov functions as far back as 1994, see [13], where the search for polynomial Lyapunov functions proving the stability of uncertain linear systems is shown to be computationally manageable via convex optimization. Moreover, in [14], the robust stability of linear systems subject to constant parametric uncertainties is studied using polynomial parameter-dependent Lyapunov functions, which are quadratic in the states and higher order homogeneous in the parameters. Presently, one of the best-known methods for computing nonquadratic Lyapunov functions and various stability, performance, and robustness guarantees comes from applying the sum-of-squares (SOS) optimization framework [15, 16] to control system problems, such as described in [17]. The SOS optimization framework provides a powerful set of tools for broad classes of systems and control verification problems. An alternative approach that leads to similar results for system analysis problems comes from applying the generalized moment problem framework [18–20], which leverages the Liouville equations used to obtain the infinite-dimensional linear differential equations that drive the evolution of probability density functions, in a way similar to the Chapman-Kolmogorov equations for Markov chains [21, Chapter 16]. This SOS-moment hierarchy fashions its search for guarantees in a manner similar to the SOS optimization-based search for dual Lyapunov-like certificates.

The work presented here streamlines that of Zelentovsky [13] by introducing, for a given linear, time-varying system, a hierarchy of related systems with increasingly high state-space dimension, whose stability analysis by means of quadratic Lyapunov functions leads directly to the computation of polynomial Lyapunov functions for the original system. While the search for quadratic Lyapunov functions for these higher-order systems is shown to be equivalent to the computation of polynomial Lyapunov functions using SOS techniques, we believe the present work leads the reader to streamlined and intuitive analysis procedures relying only on elementary linear systems theory and the knowledge of convex optimization procedures, which the authors believe is the core contribution of this paper.

The paper then extends the use of the foregoing framework to address related problems. First, the computation of reachable sets from a given initial condition for uncertain, linear time-varying systems is discussed. The ability for invariant and homogeneous polynomials to capture such reachable sets is illustrated on a second-order example for which graphical illustrations are provided. Another set of applications of the proposed framework is then considered, where the system under consideration can be as simple as a single linear, time-invariant system, and its pointwise-in-time performance is the subject of concern. More precisely, we look at using the framework of this paper to compute bounds on impulse and step responses, and we show that the framework of the paper allows the analyst to develop much more accurate bounds than can be obtained from quadratic Lyapunov functions alone.

The paper then proceeds with the computation of stability margins for the systems under consideration, which is the natural complement to the stability analyses discussed above. The problem under consideration is the continuous-time equivalent of computing the joint spectral radius, which can be thought of as the worst-case norm of an infinite product of matrices chosen from a finite set as addressed in [22] by using a Kronecker product-based lifting of the matrix set and improved upon in [23] by computing a certificate of contractibility using SOS optimization. In Section V, theory from [24], which characterizes the set of asymptotically stable trajectories for an LTV system, is leveraged in order to find periodic trajectories and bound the system's stability margin using homogeneous polynomial Lyapunov functions. Supporting the computation of upper and lower bounds on stability margins, the paper also focuses on computing "worst-case trajectories", by computing trajectories that make the computed Lyapunov functions decay as slowly as possible.

Last, the concepts described in this paper are illustrated on the stability analysis of an aerospace test article designed and flown by the second author's team for the purpose of creating reduced gravity conditions. It is shown that relying on higher-order, homogeneous polynomial Lyapunov functions considerably extends the range of conditions under which the system, a high-performance quadrotor, can be proven closed-loop stable.

Summarizing, the contributions of this paper are

1) a lifting process leading to the formulation of the search for quadratic Lyapunov functions in larger state-spaces as an elementary mechanism to search for polynomial homogeneous Lyapunov functions in the original state-space,
2) the introduction of a large number of system performance metrics that may be easily bounded by using such quadratic Lyapunov functions,
3) the computation of stability margins for time-varying linear systems expressed in continuous time,
4) the illustration of the framework on problems of aerospace interest.

II. Notation

Denote by $\mathbb{R}$ the set of real numbers and denote by $\mathbb{R}^+$ the set of non-negative real numbers. Denote by $S^n_{++} \subset \mathbb{R}^{n\times n}$ the set of symmetric positive definite $n \times n$ matrices. For $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ means $P \in S^n_{++}$ such that the quadratic form $V(x) = x^T Px$ is positive for all nonzero $x \in \mathbb{R}^n$. Last, denote by I_n the $n \times n$ identity matrix, and by $0_n \in \mathbb{R}^n$ the zero vector in $\mathbb{R}^n$.

For matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{l\times k}$, denote by $A \otimes B \in \mathbb{R}^{nl\times mk}$ the *Kronecker product of A and B*, as given by

$$A \otimes B := \begin{bmatrix} A_{1,1}B & \cdots & A_{1,m}B \\ \vdots & \ddots & \vdots \\ A_{n,1}B & \cdots & A_{n,m}B \end{bmatrix}, \tag{1}$$

where matrix entries are given via subscript. Given $A \in \mathbb{R}^{n\times m}$ and integer $i \geq 1$, denote by $\otimes^i A \in \mathbb{R}^{n^i \times m^i}$ the i^{th}-*Kronecker power of A*, as defined recursively by

$$\begin{aligned} \otimes^1 A &:= A, \\ \otimes^i A &:= A \otimes (\otimes^{i-1} A), \qquad i \geq 2. \end{aligned} \tag{2}$$

An important property of the Kronecker product that we utilize in this work, often without referring directly to it, is

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{3}$$

holds for all matrices A, B, C, D with dimensions permitting the formation of products AC and BD.

The time derivative of a vector function of time $x : t \to x(t) \in \mathbb{R}^n$ is denoted $\dot{x} : t \to \frac{d}{dt}x(t) \in \mathbb{R}^n$, and we extend this notation to matrices so that $\dot{A} \in \mathbb{R}^{n\times m}$ denotes the time-derivative of $A(t)$. Last, for a set $\mathcal{M} \subset \mathbb{R}^{n\times n}$, denote by

$\mathbf{conv}(\mathcal{M}) \subset \mathbb{R}^{n\times n}$ the *convex hull of* $\mathcal{M}$, which is the intersection of all convex subsets of $\mathbb{R}^{n\times n}$ which contain $\mathcal{M}$.

III. Stability of Linear Time-Varying Systems

A. Preliminaries

Consider the linear time-varying system

$$\dot{x} = A(t)x, \tag{4}$$

where $x := x(t) \in \mathbb{R}^n$ is the system state. We assume throughout this work that $A(t) \in \mathbb{R}^{n\times n}$ evolves inside a set $A(t) \in \mathbf{conv}(\mathcal{M}) \subset \mathbb{R}^{n\times n}$ for all $t \geq 0$, where

$$\mathcal{M} := \{A_1, A_2, \ldots, A_N\}. \tag{5}$$

The system (4) is studied in the context of Lyapunov's stability theory. A common Lyapunov function is a sufficiently smooth mapping $V : \mathbb{R}^n \rightarrow \mathbb{R}$ with $V(0) = 0$ such that for all $x \in \mathbb{R}^n\backslash\{0\}$ and for $j \in \{1, \cdots, N\}$ we have

$$V(x) > 0 \quad \text{and} \quad \dot{V}(x) = \langle \frac{\partial V}{\partial x}, A_j x\rangle \leq 0. \tag{6}$$

The system (4) is *globally stable* if and only if there exists a radially unbounded $V(x)$ satisfying (6), and the system is *globally asymptotically stable* if we also have

$$\dot{V}(x) = \langle \frac{\partial V}{\partial x}, A_j x\rangle < 0 \qquad \forall x \neq 0. \tag{7}$$

In [25, Theorem 4.5] and [26] it is shown that stability and asymptotic stability (we often neglect use of the word *global*, but any stability property for systems studied in this paper holds globally) for (4) are each equivalent to the existence of a homogeneous polynomial Lyapunov function $V(x)$ that satisfies (6) and (7) for each respective case.

When there exists a $V(x)$ satisfying (6) or (7) that is quadratic in the entries of x, we say that the system (4) is *quadratically stable* or *quadratically asymptotically stable*, respectively [27]. Such a Lyapunov function will take the form $V(x) = x^T P x$ where $P \in S^n_{++}$. Then (6) becomes

$$A_j^T P + P A_j \preceq 0 \tag{8}$$

and (7) becomes

$$A_j^T P + P A_j \prec 0 \tag{9}$$

for all $j \in \{1, \cdots, N\}$. Quadratic Lyapunov functions are the simplest instantiation of homogeneous polynomial

Lyapunov functions. Hence, the search for a quadratic Lyapunov function for (4) has computational advantages in comparison to other strategies for stability analysis [28]. The search can be reduced to solving a convex feasibility problem involving linear matrix inequalities (LMI), and many efficient solvers exist to solve such problems.

The equivalence of stability and quadratic stability for LTI systems is a well-known result [29, Chapter 4]. If the system (4) is time-invariant, *i.e.*, $N = 1$, then (4) is stable if and only if there exists a matrix P satisfying (8). This is not true, however, in the general setting of $N \geq 2$; indeed, stable systems exist for which there is no quadratic Lyapunov function [30, Section 3]. For this reason, we establish tools to compute a homogeneous polynomial Lyapunov function which can prove stability in the general setting of (4). We next present the time-varying Lyapunov differential equation

$$\dot{X} = A(t)X + XA(t)^T, \tag{10}$$

where $X := X(t) \in \mathbb{R}^{n\times n}$ is the state and $A(t)$ retains its definition from (4).

Proposition 1. *The Lyapunov differential equation* (10) *is asymptotically stable if and only if the system* (4) *is asymptotically stable.*

Proof. ($\Rightarrow$) Assume the system (10) is asymptotically stable. Let $x(t)$ be the solution of (4) starting from any initial condition $x(0) = x_0$. Define $X(t) = x(t)x(t)^T$. Then

$$\begin{aligned}\dot{X} &= \dot{x}x^T + x\dot{x}^T \\ &= A(t)xx^T + xx^T A(t)^T \\ &= A(t)X + XA(t)^T.\end{aligned}$$

It follows that $X(t)$ converges to zero by the assumed asymptotic stability of (10). Hence $x(t)$ converges to zero for any initial condition ans it follows that (4) is globally asymptotic stable.

($\Leftarrow$) Assume the system (4) is stable, and define $X(t) \in \mathbb{R}^{n\times n}$ with initial condition $X(0) = X_0$. Any matrix can be written as the sum of diads; therefore, there exist $p_{1,0}, \cdots, p_{N,0}, q_{1,0}, \cdots, q_{N,0} \in \mathbb{R}^n$ such that

$$X_0 = \sum_{j=1}^{N} p_{j,0} q_{j,0}^T.$$

Next, consider the $2N$ trajectories that satisfy

$$\dot{p}_j = A(t)p_j, \quad p_j(0) = p_{j,0}, \quad \dot{q}_j = A(t)q_j, \quad q_j(0) = q_{j,0} \tag{11}$$

where $j \in \{1, \cdots, N\}$, and note that if (4) is asymptotically stable then (11) converges to zero.

Taking $X = \sum_{j=1}^{N} p_j q_j^T$ then yields

$$\dot{X} = \sum_{j=1}^{N} \left(\dot{p}_j q_j^T + p_j \dot{q}_j^T \right) = A(t) \left(\sum_{j=1}^{N} p_j q_j^T \right) + \left(\sum_{j=1}^{N} p_j q_j^T \right) A(t)^T = A(t)X + XA(t)^T.$$

$X = \sum_{j=1}^{N} p_j q_j^T$ is the (unique) solution to the differential equation (10) with initial condition X_0, and p_j and q_j converge to zero for all $j \in \{1, \cdots, N\}$. Therefore, $X(t)$ also converges to zero along trajectories of (10). □

Remark 1. *The proposition proves that asymptotic stability is equivalent for systems* (4) *and* (10)*, but equivalence applies to stability as well.*

In the following section we build on (10) to create a hierarchy of Lyapunov differential equations for the system (4), and stability of a system at one level in the hierarchy is shown to be equivalent to stability of a system at all levels in the hierarchy.

B. Establishing the Hierarchy of LTV Systems

The system hierarchy is built by leveraging the equivalent stability properties of (4) and (10) as presented in Proposition 1. The Lyapunov differential equation (10) can be rewritten as

$$\dot{\vec{X}} = \mathcal{A}(t)\vec{X} \tag{12}$$

by taking $\vec{X}$ to be the vectorization of X, i.e. $\vec{X} = \text{vec}(X) \in \mathbb{R}^{n^2}$. In this case, $\mathcal{A}(t) \in \mathbb{R}^{n^2 \times n^2}$ evolves nondeterministically in the set $\mathcal{A}(t) \in \{\mathcal{A}_1, \cdots, \mathcal{A}_N\}$ where, for $j \in \{1, \cdots, N\}$, we define $\mathcal{A}_j := I_n \otimes A_j + A_j \otimes I_n$. Using the Kronecker product to define $\mathcal{A}_j$ and letting $\vec{X} = \text{vec}(xx^T) = x \otimes x$, as in the proof of Proposition 1, means that (12) contains redundant expressions; for the case when $n = 2$, a simple procedure for eliminating these redundancies is given in [31].

We refer to (12), which is also linear time-varying, as the *lifted* or *augmented* system relative to system (4). Applying concepts of quadratic stability to the lifted-system, the system (12) is stable if there exists a positive definite $P \in \mathbb{R}^{n^2 \times n^2}$ so that

$$\mathcal{A}_j^T P + P\mathcal{A}_j \preceq 0, \tag{13}$$

for all $j \in \{1, \cdots, N\}$. These constraints correspond to the existence of a Lyapunov function $\mathcal{V}(\vec{X}) = \vec{X}^T P \vec{X}$ for (12), which is quadratic in the entries of $\vec{X}$. The following proposition relates the quadratic stability of a system and its augmentation.

Proposition 2. *If the system* (4) *is quadratically stable, then the system* (12) *is also quadratically stable. In particular, if* $V(x) = x^T P_1 x$ *is a quadratic Lyapunov function for* (4)*, then* $\mathcal{V}(\vec{X}) = \vec{X}^T (P_1 \otimes P_1) \vec{X}$ *is a quadratic Lyapunov function for* (12)*, i.e.,* $P_2 = P_1 \otimes P_1$ *solves* (13).

It is of course possible to repeat the process by constructing the vectorization of the Lyapunov differential equation of (12), which would be

$$\dot{\xi} = (I_{n^2} \otimes \mathcal{A}(t) + \mathcal{A}(t) \otimes I_{n^2})\xi, \tag{14}$$

with $\xi \in \mathbb{R}^{n^4}$. By finding a quadratic Lyapunov function for this new system and applying Proposition 1 twice, we see that the quadratic stability of (14) establishes the stability of both (12) and (4). It is therefore possible to construct a "hierarchy" of Lyapunov differential equations whose state space dimensions are n^l, $l = 2^{(i-1)}$, where i is an integer greater than or equal to 1 and where $i = 1$ gives the dimension of the "base" level system (4). The following proposition formalizes the relationship between a lifted system of dimension n^l and the system (4).

Proposition 3. *System* (4) *is stable if there exists* $i \in \mathbb{N}_{\geq 1}$ *and* $P_i \in \mathbb{R}^{n^l \times n^l}$, $l = 2^{(i-1)}$, *positive definite such that*

$$(\mathcal{A}_j^i)^T P_i + P_i \mathcal{A}_j^i \leq 0 \tag{15}$$

for all $j \in \{1, \cdots, N\}$, *where*

$$\begin{aligned} \mathcal{A}_j^i &:= I_{n^{2(i-2)}} \otimes \mathcal{A}_j^i + \mathcal{A}_j^i \otimes I_{n^{2(i-2)}}, \quad i \geq 2 \\ \mathcal{A}_j^1 &= A_j. \end{aligned} \tag{16}$$

Building the hierarchy using the Lyapunov differential equation by applying the recursive operation given by (16) quickly produces systems of dimension n^l. A more general system hierarchy can be constructed, where the system dimension at level i is n^i. Taking (4) as the system H_1 with state $\xi_1 = x$, we build the hierarchy H according to

$$H_1 : \begin{cases} \dot{\xi}_1 &= \mathcal{A}_1(t)\xi_1 \\ \mathcal{A}_1(t) &\in \mathbf{conv}(\mathcal{M}_1) \\ \mathcal{M}_1 &= \{\mathcal{A}_1^1, \cdots, \mathcal{A}_N^1\} \\ \mathcal{A}_j^1 &= A_j \end{cases} \qquad H_i : \begin{cases} \dot{\xi}_i &= \mathcal{A}_i(t)\xi_i \\ \mathcal{A}_i(t) &\in \mathbf{conv}(\mathcal{M}_i) \\ \mathcal{M}_i &= \{\mathcal{A}_1^i, \cdots, \mathcal{A}_N^i\} \\ \mathcal{A}_j^i &= I_n \otimes \mathcal{A}_j^{i-1} + A_j \otimes I_{n^{i-1}} \end{cases} \tag{17}$$

The system at level H_i has dimension $n^i \times n^i$. Again, constructing system H_i from H_1 is described as *lifting* or *augmenting*, and H_i is referred to as the lifted or augmented system.

Proposition 4. *If* x *is a trajectory of* (4), *then* ξ_i *is a trajectory of* H_i *when* $\xi_i = \otimes^i x$.

Proof. We first show that if $A(t) = \theta_1(t)A_1 + \cdots + \theta_N(t)A_N$ for scalar functions $\theta_j(t) \in [0, 1]$ and $\sum_j \theta_j(t) = 1$ for all t with $j = 1, \ldots, N$, then $\mathcal{A}^i(t) = \theta_1(t)\mathcal{A}_1^i + \cdots + \theta_N(t)\mathcal{A}_N^i$. This follows from the bilinearity of the Kronecker product, and only the inductive step is shown:

$$\begin{aligned}\mathcal{A}^i(t) &= I_n \otimes (\theta_1(t)\mathcal{A}_1^{i-1} + \cdots + \theta_N(t)\mathcal{A}_N^{i-1}) + (\theta_1(t)A_1 + \cdots + \theta_N(t)A_N) \otimes I_{n^{i-1}} \\ &= I_n \otimes \theta_1(t)\mathcal{A}_1^{i-1} + \cdots + \theta_N(t)\mathcal{A}_N^{i-1} + \theta_1(t)A_1 \otimes I_{n^{i-1}} + \cdots + \theta_N(t)A_N \otimes I_{n^{i-1}} \\ &= \theta_1(t)\mathcal{A}_1^i + \cdots + \theta_N(t)\mathcal{A}_N^i.\end{aligned}$$

Now let $\xi_i = \otimes^i x$. Then

$$\begin{aligned}\dot{\xi}_i &= \frac{d}{dt}(\otimes^i x) = (\frac{d}{dt}x) \otimes (\otimes^{i-1}x) + x \otimes \frac{d}{dt}(\otimes^{i-1}x) \\ &= A(t)x \otimes \xi_{i-1} + x \otimes \mathcal{A}^{i-1}(t)\xi_{i-1} \\ &= (A(t) \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}^{i-1}(t))\xi_i \\ &= \mathcal{A}^i(t)\xi_i.\end{aligned}$$

Therefore, $\xi_i(t)$ solves H_i. □

The following two theorems generalize Propositions 2 and 3 for the hierarchy H.

Theorem 1. *If system* (4) *is quadratically stable, then for every $i \geq 1$ there exists a quadratic Lyapunov function which proves stability for the system H_i. Moreover, if P_1 satisfies* (8)*, then $P_i = \otimes^i P_1$ satisfies*

$$(\mathcal{A}_j^i)^T P_i + P_i \mathcal{A}_j^i \preceq 0 \tag{18}$$

for the system H_i.

Proof. Without loss of generality, let $\mathcal{M}_1 = \{A\}$. Then

$$\begin{aligned}(\mathcal{A}^i)^T P_i &= (I_n \otimes \mathcal{A}_j^{i-1} + A_j \otimes I_{n^{i-1}})^T (P_1 \otimes P_{i-1}) \\ &= P_1 \otimes (\mathcal{A}^{i-1})^T P_{i-1} + A^T P_1 \otimes P_{i-1}, \\ P_i \mathcal{A}^i &= (P_1 \otimes P_{i-1})(I_n \otimes \mathcal{A}^{i-1} + A \otimes I_{n^{i-1}}) \\ &= P_1 \otimes P_{i-1}\mathcal{A}^{i-1} + P_1 A \otimes P_{i-1}.\end{aligned}$$

Then the stability requirement (18) is

$$(\mathcal{A}^i)^T P_i + P_i \mathcal{A}^i = P_1 \otimes ((\mathcal{A}^{i-1})^T P_{i-1} + P_{i-1}\mathcal{A}^{i-1}) + (A^T P_1 + P_1 A) \otimes P_{i-1}. \tag{19}$$

For two square matrices L and M of dimensions l and m with eigenvalues $\lambda_j,\ j = 1, \ldots, l$ and $\mu_k,\ k = 1, \ldots, m$

respectively, the eigenvalues of $L \otimes M$ are

$$\lambda_j \mu_k, \quad j = 1, \ldots, l, \ k = 1, \ldots, m.$$

Additionally, $(L \otimes M)^T = L^T \otimes M^T$. If L is negative-(semi)definite and M positive-(semi)definite, $L \otimes M$ is negative-(semi)definite. Therefore, $(A^T P_1 + P_1 A) \otimes P_{i-1} \preceq 0$ since P_{i-1} is positive-definite. It can easily be shown by induction that the term $P_1 \otimes ((\mathcal{A}^{i-1})^T P_{i-1} + P_{i-1}\mathcal{A}^{i-1})$ is also negative-semidefinite; therefore, (18) is satisfied with $P_i = \otimes^i P_1$. □

Theorem 2. *System* (4) *is asymptotically stable if there exists $i \in \mathbb{N}_{\geq 1}$ and $P_i \in S_{++}^{n^i}$ such that $(\mathcal{A}_j^i)^T P_i + P_i \mathcal{A}_j^i \prec 0$ is satisfied for all $j \in \{1, \cdots, N\}$, where A_j^i is given by* (17).

Proof. Assume that, for some i, there exists a P_i which satisfies the inequality in the theorem, proving that the system H_i is asymptotically stable. Take $\xi_i = \otimes^i x$. Proposition 4 shows that $\dot{\xi}_i = \mathcal{A}(t)\xi_i$. Then $\lim_{t\to\infty} \xi_i(t) = 0_{n^i} \implies \lim_{t\to\infty} x(t) = 0_n$. □

Next comes an important theorem, because it states that the approach described in this paper is not more, nor less powerful than "Sum-of-Squares" methods as far as stability analyses are concerned.

Theorem 3. *The system* (4) *is proven stable via the conditions stated in Theorem 2 if and only if it is proven stable by means of a Sum-of-Squares Lyapunov function as described in [25] and [26].*

Proof. The proof is available in [31] and will not be repeated here. □

Remark 2. *As with Proposition 1, Theorem 2 can also be easily extended to the case of stability (rather than asymptotic stability) when P_i satisfies* (18).

The following corollary will be used extensively to produce Lyapunov functions for the applications and examples in the remainder of the paper.

Corollary 1. *Given a P_i which satisfies* (18) *for system H_i, the function*

$$V(x) = \xi_i^T P_i \xi_i \tag{20}$$

with $\xi_i = \otimes^i x$ is a homogeneous polynomial Lyapunov function for H_1 of order $2i$.

IV. Applications in System Analysis for Linear Time Varying Systems

A. Stability Analysis and Construction of Invariant Sets

When the switching or time-varying nature of a system is nondeterministic, many possible trajectories can result from a given initial condition. Therefore, bounds on the set of possible trajectories is of interest, even when the system is stable.

Definition 1 (Invariant Set)**.** *A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be invariant w.r.t* (4) *if for every trajectory that starts with initial condition $x(0) \in \mathcal{X}$ and under the dynamics* (4)*, $x(t) \in \mathcal{X}$ for all $t > 0$.*

Proposition 5. *For a $P_i \in S_{++}^{n^i}$ that satisfies* (18) *for some $i \geq 1$ and for all $\mathcal{A}_j^i \in \mathcal{M}_i$, the set*

$$\mathcal{X} = \{x \mid \xi_i^T P_i \xi_i \leq \xi_i(0)^T P_i \xi_i(0),\ \xi_i = \otimes^i x\} \tag{21}$$

is invariant under (4).

Remark 3. *Proposition 5 presents a convex feasibility problem which provides a sufficient condition for constructing the invariant set $\mathcal{X}$; however, a measure of this set's "quality" is subjective. Generally, some kind of set optimality is sought. A set which fits tightly around possible trajectories can usually be obtained by minimizing a convex objective function such as* $\text{trace}(P)$ *or* $\log(\det(P^{-1}))$.

A numerical example is presented which illustrates that higher levels i of the hierarchy can significantly improve the invariant set produced using Proposition 5.

Example 1. *Consider the system* (4) *with $\mathcal{M}_1 = \{A_1, A_2\}$ and*

$$A_1 = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.5 & 2.5 \\ -2.5 & 1.5 \end{bmatrix}. \tag{22}$$

The system is stable, and forward invariant regions are computed by applying Proposition 5 with quadratic Lyapunov functions for four systems in the hierarchy (17) *using levels $i = 1, 5, 8,$ and 13. Consider the initial condition $x_0 = [1,\ 0]^T$. These Lyapunov functions are computed using the convex program*

$$\begin{aligned} &\textit{minimize} \ \otimes^i x_0^T P_i \otimes^i x_0 = P_i(1,1), \\ &\textit{subject to } (18) \textit{ and } P_i \geq I_{n^i} \end{aligned}$$

for $j = 1$ and 2, where the objective function is added to favor tight invariant regions. The results of this example are plotted graphically in Figure 1. The convex programs in this example and those that follow are computed using the

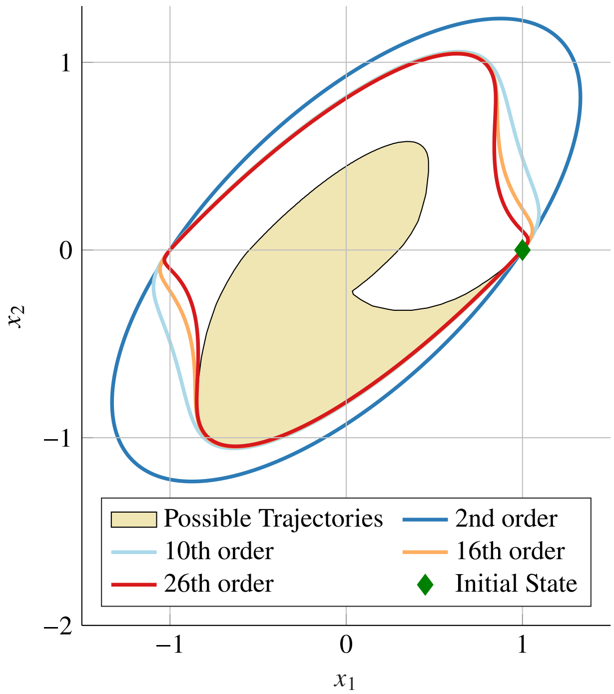

Fig. 1 Simulated system response of (4) with parameters given by (22). With initial condition $x_0 = [1,\ 0]^T$, the system can only reach the region shown in light yellow, which was computed via simulation. The dark blue, light blue, orange and red regions represent invariant sets calculated using polynomial homogeneous Lyapunov functions of degree 2, 10, 16, and 26, respectively.

SDPT3 solver supported by CVX [32], a convex optimization toolbox made for use with MATLAB. As the order of the homogeneous Lyapunov functions increase, the Lyapunov functions level sets appear to converge to the union of the possible trajectories from the initial condition $[1, 0]^T$ *and its symmetric about the origin* $[-1, 0]^T$ *as shown in Fig. 2. Clarifying this conjecture is left for later work.*

Homogeneous higher order Lyapunov functions can be used to capture better and more accurate bounds for peak norms of the LTI systems. The bounds obtained using higher order Lyapunov functions also reduces the conservatism of the bounds resulted from using quadratic Lyapunov functions.

In the next section, the hierarchy of quadratic lyapunov functions is used to analyze and get accurate bounds for the impulse and step responses for the class of LTI systems.

B. Pointwise-in-Time Analysis

Classical analyses of LTI and LTV system impulse- and step-responses can be improved by lifting the system using the hierarchy (17). We investigate peak-response properties of the system

$$\dot{x} = A(t)x + bu, \quad y = cx \tag{23}$$

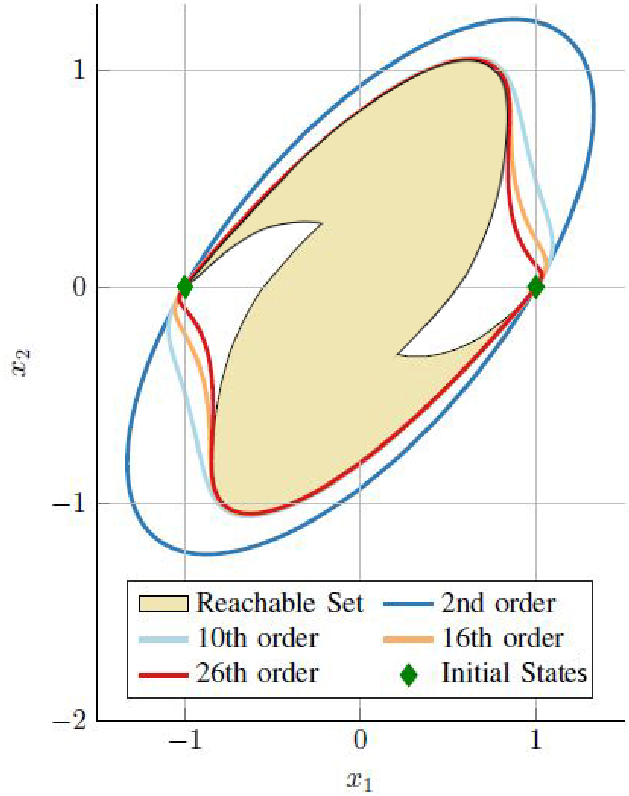

Fig. 2 Simulated system response of (4) with parameters given by (22). With initial conditions $x_0 \in \{[1,\ 0]^T,\ [-1,\ 0]^T\}$ the system can only reach the symmetric reachable set shown in light yellow. As the degree of the computed invariant sets go up, they appear to converge to this reachable set.

with scalar input $u = u(t)$, $b \in \mathbb{R}^n$, and $c^T \in \mathbb{R}^n$. $A(t) \in \mathcal{M}_1$, as with the system (4).

Proposition 6. *The impulse response of* (23)*, denoted by $h(t)$, is given by the output of the following LTV system [33]:*

$$\dot{\varphi} = A(t)\varphi, \quad h(t) = c\varphi(t) \quad \varphi(0) = b. \tag{24}$$

The proposition allows us to study the impulse response of (23) by properly augmenting the system (24) with the hierarchy (17). Define $\mathbf{b}_i \in \mathbb{R}^{n^i}$ and $\mathbf{c}_i^T \in \mathbb{R}^{n^i}$ by

$$\mathbf{b}_i = \otimes^i b, \qquad \mathbf{c}_i = \otimes^i c \tag{25}$$

for an integer $i \geq 1$. The system (24) when lifted to H_i in the hierarchy has $\xi_i(0) = \mathbf{b}_i$ and output $\mathbf{h}_i(t) = \mathbf{c}_i\xi_i(t) = h(t)^i$.

Theorem 4. *The impulse response of* (23) *satisfies $|h(t)| \leq \overline{h}$ for all t, where*

$$\bar{h} = (\mathbf{c}_i P_i^{-1}\mathbf{c}_i^T)^{1/(2i)}(\mathbf{b}_i^T P_i \mathbf{b}_i)^{1/(2i)} \tag{26}$$

for any $P_i \in S_{++}^{n^i}$ *that satisfies* (18).

Proof. The optimization problem

$$\overline{\mathbf{h}}_i = \max_{\xi_i \in \mathbb{R}^{n^i}} \mathbf{c}_i \xi_i, \qquad \text{subject to } \xi_i^T P_i \xi_i = \mathbf{b}_i^T P_i \mathbf{b}_i \tag{27}$$

finds the point on the boundary of the set $\{\xi_i \mid \xi_i^T P_i \xi_i \leq \mathbf{b}_i^T P_i \mathbf{b}_i\}$ that is in the direction $\pm\mathbf{c}_i$. The set is invariant under the dynamics H_i since P_i satisfies (18). The optimization problem is solved by

$$\overline{\mathbf{h}}_i = (\mathbf{c}_i P_i^{-1} \mathbf{c}_i)^{1/2} (\mathbf{b}_i^T P_i \mathbf{b}_i)^{1/2}.$$

Since $\mathbf{h}_i(t) = h(t)^i$, we have $|h(t)|^i = |\mathbf{h}(t)| \leq \overline{\mathbf{h}}_i^{(1/i)} = \overline{h}$, which is therefore given by (26). □

Remark 4. *Theorem 4, as well as those that follow, can utilize any Lyapunov parameter* P_i *which satisfies* (18). *The results can be improved by computing* P_i *via the following convex program,*

$$\begin{aligned} P_i = \quad & \underset{Q \in S_{++}^{n^i}}{\arg\min} \quad \mathbf{c}_i Q^{-1} \mathbf{c}_i{}^T \\ & s.t. \quad \mathbf{b}_i{}^T Q \mathbf{b}_i \leq 1 \\ & \qquad\; (\mathcal{A}_j^i)^T Q + Q \mathcal{A}_j^i \leq 0, \end{aligned} \tag{28}$$

which computes a P_i *that minimizes* $\overline{\mathbf{h}}_i$ *while adhering to the stability constraints.*

Example 2. *Theorem 4 is particularly useful for systems with stiff dynamics. Consider the system* (23) *with*

$$\begin{aligned} A &= \begin{cases} -(m)^{k-1} & \textit{for diagonal entry in row } k = 1, \ldots, n \\ 0 & \textit{otherwise,} \end{cases} \\ b &= [1, \ldots, 1]^T, \\ c &= \begin{cases} 1 & \textit{for first entry} \\ 2(-1)^{k+1} & \textit{for entries } k = 2, \ldots, n \end{cases} \end{aligned} \tag{29}$$

for some $m >> 1$*. This system is stable, and* $|h(t)| \leq 1$ *for all* $t \geq 0$*. However, it was shown in [34] that the gap between the actual maximum impulse response and the upper bound obtained with a quadratic Lyapunov function grows to* $2n - 1$ *when* m *tends toward infinity. With* $n = 2$ *and* $m = 100$*, we study the impulse response by using* (26) *with* $i = 1,\ 2,$ *and*

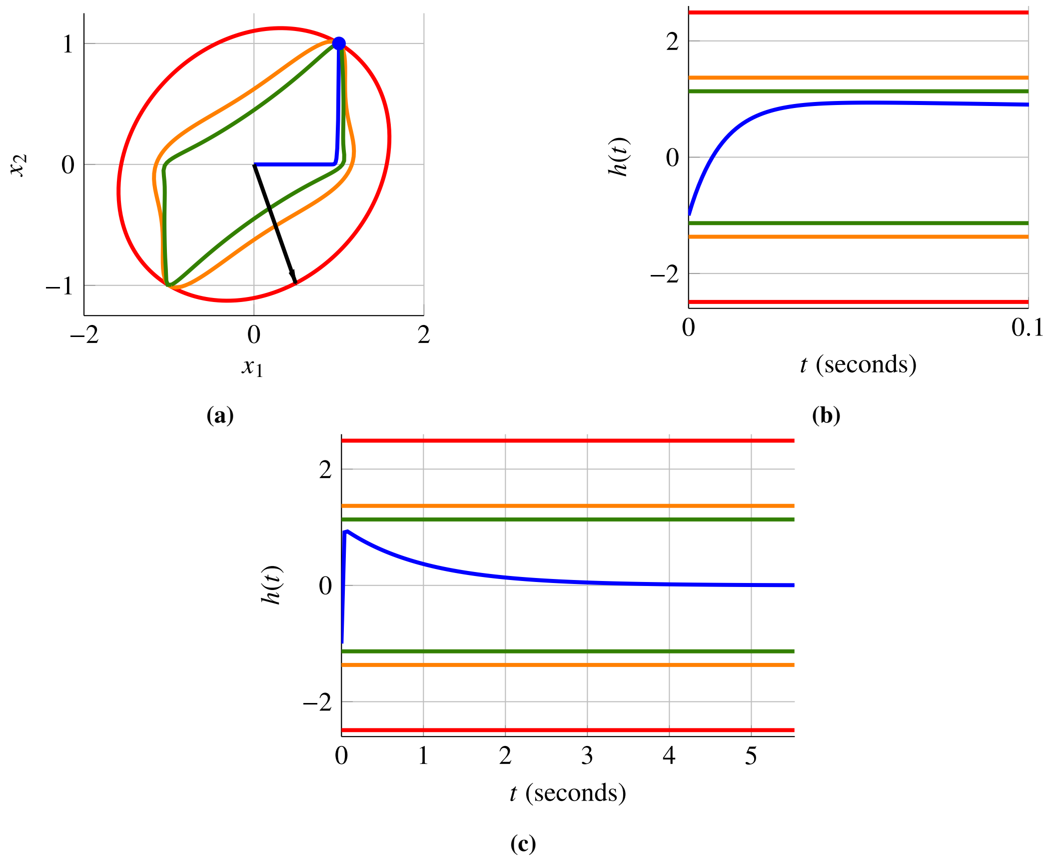

Fig. 3 Example 2. (a) Phase portrait of the impulse response for the stiff system from Example 2 where $n = 2, m = 100$. The impulse response is shown in blue, and $x(0) = b$ is shown as a blue dot. A vector in the direction of c^T is shown in black. The invariant level sets of the 2^{nd}, 4^{th} and 10^{th} order homogeneous polynomial Lyapunov functions that are computed in this study are shown in red, orange and green, respectively. (b) The impulse response is shown in blue, and the magnitude bounds derived using P_i for $i = 1,\ 2,$ and 5 are shown in red, orange and green, respectively. (c) The same impulse response is plotted over a longer time horizon, where it can be seen that $h(t)$ decays to 0.

5, where P_i is computed with (28). *Figure 3 shows that increasing the level of the hierarchy can significantly improve the impulse response bound for this system.*

Next, a time-dependent bound on the impulse response of (23) is sought.

Theorem 5. *For $\alpha \in \mathbb{R}$, if $P_i \in S_{++}^{n^i}$ satisfies* (18) *for some i using the shifted set $\overline{\mathcal{M}}_1 = \{A_j + \alpha I_n \mid A_j \in \mathcal{M}_1\}$ in (17), then the impulse response of* (23) *satisfies $|h(t)| \leq e^{-\alpha t}\overline{h}$ for all t, where $\overline{h}$ is given by* (26).

Proof. Consider the system

$$\dot{\varphi}_\alpha = (A(t) + \alpha I_n)\varphi_\alpha.$$

The solution is $\varphi_\alpha(t) = e^{\alpha t}\varphi(t)$, where $\varphi(t)$ solves (24). Therefore, $h(t) = c\varphi(t) = e^{-\alpha t}c\varphi_\alpha(t) \leq e^{-\alpha t}\overline{h}$. □

Example 3. *Consider the uncertain system* (23) *where $\mathcal{M}_1 = \{A - \Delta,\ A + \Delta\}$ with system parameters*

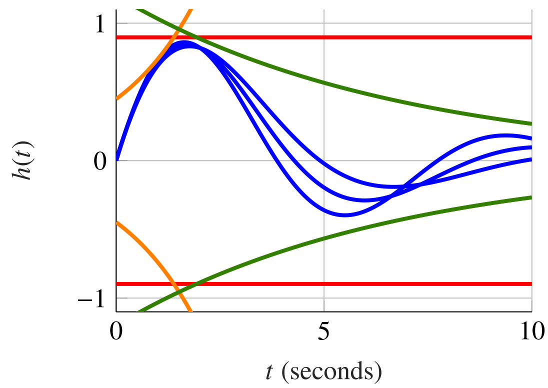

Fig. 4 Example 3. Three impulse response simulations are conducted, plotted in blue. A global norm bound on $h(t)$ is computed using Theorem 4 for $i = 6$ and is shown in red. Two exponential bounds on $h(t)$ are constructed from Theorem 5 with $\alpha = -0.5,\ 0.15$ and $i = 6$, and the bounds are shown in orange and green, respectively. A complete envelope can be obtained by combining the three bounding curves.

$$A = \begin{bmatrix} 0 & 1 \\ -0.6 & -0.5 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & 0 \\ 0.1 & -0.1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{30}$$

The optimization problem (28) *is solved for* $i = 6$*, and the resulting Lyapunov parameters* P_6 *are used to generate bounds on the impulse response using* (26)*. Next, Theorem 5 is employed using* $i = 6$*, and exponential bounds are generated for* $\alpha = -0.5$ *and* 0.15*. The exponential and global norm bounds are plotted in Figure 4 alongside several sample impulse responses. Applying the hierarchy and using higher-order Lyapunov functions provides some improvement over using quadratic Lyapunov functions in this example. For instance, when* $\alpha = 0$*,* $\bar{h} = 0.99$ *when using a quadratic Lyapunov function, and* $\bar{h} = 0.90$ *with a* 12^{th} *order Lyapunov function. A tighter envelope is therefore readily available for the system impulse response.*

An alternative method for studying the impulse response of an uncertain system involves comparing it to a system whose impulse response is known. Consider the system

$$\dot{\widetilde{x}} = \begin{bmatrix} A(t) & 0 \\ 0 & A \end{bmatrix} \widetilde{x} + \begin{bmatrix} b \\ b \end{bmatrix} u, \quad y = \begin{bmatrix} c & -c \end{bmatrix} \widetilde{x}, \tag{31}$$

with impulse response $\widetilde{h}(t) = h(t) - ce^{At}b$, where $h(t)$ is the impulse response of (23), which is considered to be a "subsystem" of (31). A new time-varying bound on $h(t)$ is derived using the previous results. Define $\overline{A}_j \in \mathbb{R}^{2n\times 2n}$, and $\overline{\mathbf{b}}_i$, $\overline{\mathbf{c}}_i^T \in \mathbb{R}^{(2n)^i}$ by

$$\overline{A}_j = \begin{bmatrix} A_j & 0 \\ 0 & A \end{bmatrix} \text{ with } A_j \in \mathcal{M}_1, \quad \overline{\mathbf{b}}_i = \otimes^i \begin{bmatrix} b \\ b \end{bmatrix}, \quad \overline{\mathbf{c}}_i = \otimes^i \begin{bmatrix} c & -c \end{bmatrix}. \tag{32}$$

Theorem 6. *For $\alpha \in \mathbb{R}$, if $P_i \in S_{++}^{(2n)^i}$ satisfies* (18) *for some i with the shifted set $\overline{\mathcal{M}}_1 = \{\overline{A}_j + \alpha I_{2n} \mid A_j \in \mathcal{M}_1\}$, then*

$$|h(t) - ce^{At}b| \leq e^{-\alpha t}\overline{\mathbf{h}} \tag{33}$$

for all t where $\overline{\mathbf{h}}$ is given by

$$\overline{\mathbf{h}} = (\overline{\mathbf{c}}_i P_i^{-1} \overline{\mathbf{c}}_i^T)^{1/(2i)} (\overline{\mathbf{b}}_i^T P_i \overline{\mathbf{b}}_i)^{1/(2i)}. \tag{34}$$

Moreover, the parameter P_i which minimizes $\overline{\mathbf{h}}$ can be computed with (28) *by using $\overline{\mathcal{A}}_j^i \in \overline{\mathcal{M}}_i$, $\overline{\mathbf{b}}_i$, $\overline{\mathbf{c}}_i$, and decision variables $Q \in S_{++}^{(2n)^i}$.*

The proof follows directly from applying Theorem 5 to system (31).

Remark 5. *The analysis in Theorem 6 is observed to provide a tighter bound when A is chosen to be the centroid of $\mathcal{M}_1$.*

We next compute a bound on the step response of (23) when it is LTI, as in, $A(t) \equiv A$. The step response (where $u = 1$) is given by

$$s(t) = cA^{-1}(e^{At} - I_n)b. \tag{35}$$

where $x(0) = 0_n$. In the following theorem, we will use the notation $\mathbf{A}_i^{-1} = \otimes^i(A^{-1})$ and $\overline{\mathbf{b}}_i = \otimes^i(A^{-1}b)$.

Theorem 7. *If $P_i \in S_{++}^{n^i}$ satisfies* (18) *for some i, then the step response of* (23) *is bounded such that $|s(t) + cA^{-1}b| \leq \overline{s}$ for all t, where*

$$\overline{s} = (\mathbf{c}_i P_i^{-1} \mathbf{c}_i^T)^{1/(2i)} (\overline{\mathbf{b}}_i^T P_i \overline{\mathbf{b}}_i)^{1/(2i)} \tag{36}$$

Proof. The step response of (23) is equivalent to the impulse response of the system

$$\dot{\bar{x}} = A\bar{x} + A^{-1}bu, \quad \bar{y} = c\bar{x} - cA^{-1}b.$$

Thus, the bound (36) is derived using Theorem 4. □

As in expression (28), a tighter bound on the step response can be computed using P_i from

$$\begin{aligned} P_i = \quad \underset{Q \in S_{++}^{n^i}}{\arg\min} \quad & \mathbf{c}_i Q^{-1} \mathbf{c}_i{}^T \\ \text{s.t.} \quad & \overline{\mathbf{b}}_i^T P_i \overline{\mathbf{b}}_i \leq 1 \\ & (\mathcal{A}^i)^T Q + Q\mathcal{A}^i \leq 0 \end{aligned} \tag{37}$$

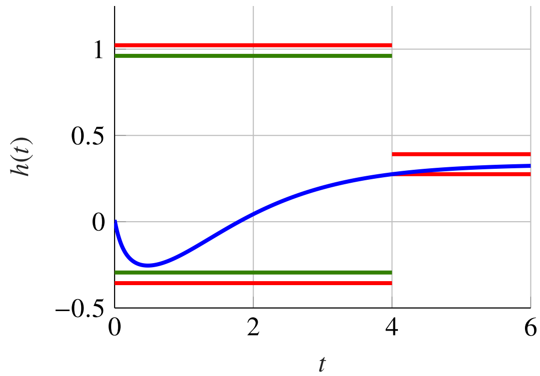

Fig. 5 Example 4. The step response $s(t)$ is shown in blue, and the magnitude bounds derived using P_i for $i = 1, 3$ are shown in red and green, respectively. At time $t = 4$, a new bound is computed via (36) using $i = 1$ and is shown in red.

Example 4. *Consider the system* (23) *with*

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -10 & 1 \\ 0 & -2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad c^T = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Solving (36) *for $i = 1$ and 3 then applying Theorem 7, a bound on the step response from $t = 0$ is produced. At time $t = 4$, a new bound on the step response is computed via* (36) *by setting $b = x(4)$, and this creates a norm bound on the tail of $s(t)$. The results are shown in Figure 5.*

V. Discovering Worst-Case Trajectories and Bounding the Stability Margin

The aforementioned analyses of (4) provide over-approximating characterizations of the systems' behavior. The use of the hierarchy (17) provides an elegant and intuitive way to reduce conservatism of analyses that are traditionally performed using quadratic Lyapunov functions. While plots of simulations of a system can provide an idea of how overapproximating a bound is, some measure of distance between a computed bound and a worst-case trajectory would provide a helpful way to bound stability and performance margins from above and from below. In this section, we present a procedure to bound the conservativeness of the analyses presented above. Computing a marginally stable or destabilizing trajectory for (4) is similar to the problem of over-approximating the joint spectral radius of a discrete-time uncertain linear system [23].

Several theorems developed by Pyatnitskiy and Rapoport [24] formalize the intuition that motivates the work in this section. The authors study the case when the system (4) can admit a trajectory where the system state never goes to zero but does not admit a trajectory that diverges. If (4) can produce this trajectory, we informally say that the system is *marginally stable* and we refer to this trajectory as a *marginally stable trajectory* (even though the system can often produce trajectories that will go to zero as well). Marginal stability for (4) is a property of the set (5) and is defined formally in [24] as well as in Section V.B for a special case of (4). In [24, Theorem 1], the authors prove that there exists a positive convex function $v(x)$ for which $\dot{v}(x) = 0$ for almost all $t > 0$ along a marginally stable trajectory. It is well known that for systems with $n = 2$ and $n = 3$, this trajectory becomes periodic. The fact that an invariant function $v(x)$ exists when the system can produce a marginally stable trajectory motivates us to approximate $v(x)$ using a high-order homogeneous polynomial Lyapunov function $V(x)$; we can then discover a case when (4) can approach marginal stability by constructing a trajectory that maximizes $\dot{V}(x)$. By perturbing (4) to the point where it is marginally stable, we can approximate a measure of the system's stability margin.

A. Computing a Worst-Case Trajectory

A worst-case trajectory is one that results from choosing an $A(t)$ for each t which "pushes" the system close to instability for a given initial condition. Such a trajectory can be defined using a Lyapunov-like function to guide the choice of $A(t)$.

Definition 2 (Worst-case trajectory)**.** *A trajectory $\phi(t; x_0)$ which solves* (4) *with $x(0) = x_0$ and $A(t)$ given by*

$$A_w(t; V) = \underset{A(t)\in\mathbf{conv}(\mathcal{M}_1)}{\arg\max} \quad \dot{V}(\phi(t; x_0)), \tag{38}$$

where $V(x)$ is a continuously differentiable function, is a worst-case trajectory.

Any $V(x)$ can produce a worst-case trajectory, but the insight provided by (38) is most enlightening for system analysis when $V(x)$ is a Lyapunov function due to the observations in [24]. A system that can produce a trajectory that evolves about the level set of a Lyapunov function is close to instability, and such a trajectory can be constructed using (38).

Proposition 7. *Given a positive-definite function $V(x) = \xi_i^T P_i \xi_i$ at level i in the hierarchy* (17)*, $A_w(t; V)$ can be computed by*

$$A_w(t; V) = \underset{A_k \in \mathcal{M}_1}{\arg\max} \quad \xi_i^T((\mathcal{A}_k^i)^T P_i + P_i \mathcal{A}_k^i)\xi_i \tag{39}$$

where $\xi_i = \otimes^i x$ solves the system H_i from (17) *with some initial condition $\xi_i(0)$.*

Proof.

$$\begin{aligned}\dot{V}(t) &= \xi_i^T(\mathcal{A}^i(t)^T P_i + P_i \mathcal{A}^i(t))\xi_i \\ &= \xi_i^T((\theta_1 \mathcal{A}_1^i + \cdots + \theta_N \mathcal{A}_N^i)^T P_i + P_i(\theta_1 \mathcal{A}_1^i + \cdots + \theta_N \mathcal{A}_N^i))\xi_i\end{aligned}$$

for $\sum_{k=1}^{N} \theta_k, \theta_k = 1$. Then

$$\dot{V}(t) = \theta_1 \left[\xi_i^T ((\mathcal{A}_1^i)^T P_i + P_i \mathcal{A}_1^i) \xi_i \right] + \cdots + \theta_N \left[\xi_i^T ((\mathcal{A}_N^i)^T P_i + P_i \mathcal{A}_N^i) \xi_i \right]$$

is maximized for some $\theta_k = 1,\ k = 1, \ldots, N$. Therefore all other $\theta_l = 0,\ l \neq k$ and we have $\mathcal{A}^i(t) = \mathcal{A}_k^i \implies A(t) = A_k$. □

Remark 6. *Proposition 7 simply amounts to selecting the $\mathcal{A}_k^i,\ k \in 1, \ldots, N$ which maximizes the expression for $\dot{V}(x)$ at each point on the trajectory $x(t)$. This can be done in practice by simulating the system from an initial condition x_0 with discrete time steps and selecting $A \in \mathcal{M}_1$ at each time step based on the maximization criteria. Whether* (4) *is a switching system with a finite set of possible system matrices $A_k \in \mathcal{M}_1$ or an LTV system with $A(t) \in$* **conv**$(\mathcal{M}_1)$, *the worst-case trajectory is the same. To see this, let each θ_k be a time-dependent function $\theta_k(t)$. Then $\dot{V}(t)$ is still maximized when $\theta_k(t) = 1$ at each t for some $k = 1, \ldots, N$.*

A worst-case trajectory is used to produce an upper bound on the conservatism of a stability analysis.

B. Bounding the Stability Margin from Above

In order to measure the conservativeness of a bound on system behavior, a notion of stability margin is needed. A special case of (4) is constructed in order to present this analysis. In this special case, $\mathcal{M}_1 = \{A, A + \delta A_0\}$ for some $\delta \geq 0$ so that

$$\begin{aligned} \dot{x} &= A(t)x \\ A(t) &= A + \Delta(t) A_0 \\ \Delta(t) &\in [0,\ \delta] \end{aligned} \tag{40}$$

We study the stability margin of (40) by perturbing a nominal system matrix A by $\Delta(t)A_0$. This formulation can encapsulate the widely-studied Lur'e system [4], which has the form

$$\begin{aligned} \dot{x} &= Ax + Bp \\ y &= Cx \\ p &= \phi(t, y) \end{aligned}$$

for a memoryless nonlinearity $\phi(t, \cdot) : \mathbb{R} \to \mathbb{R}$ which belongs to the sector $[0,\ \delta]$ for each t. In this case, $A_0 = BC$. The construction (40) suggests a straightforward definition for stability margin

Definition 3 (Stability margin)**.** *Given $A_0,\ A \in \mathbb{R}^{n \times n}$, with A Hurwitz, and $\delta \in \mathbb{R}^+$, the system* (40) *is stable with* *respect to* δ *if* (40) *is stable for all* $\Delta(t)$ *such that* $\Delta(t) \in [0,\ \delta]$ *for all* $t \geq 0$. *The stability margin for* (40) *is the unique* $\hat{\delta} \in [0, \infty]$ *such that* (40) *is stable with respect to all* $\delta \in [0,\ \hat{\delta}]$ *and such that* (40) *is not stable with respect to* $\hat{\delta} + \varepsilon$ *for any* $\varepsilon > 0$.

The stability of (40) can be studied via the eigenvalues of a discrete transition matrix that has been constructed from a worst-case trajectory, as suggested by an observation by Pyatnitskiy and Rapoport.

Proposition 8. *[24, Theorem 5]. Let* $\Phi(t, t_0)$ *be the fundamental matrix of solutions for* (40) *such that* $\Phi(t_0, t_0) = I$ *and* $x(t) = \Phi(t, t_0)x(t_0)$. *Denote the spectral radius of* $\Phi(t, t_0)$ *by* $\rho(\Phi)$ *and define*

$$\Re(\mathcal{M}_1) := \limsup_{t\to\infty} \max_{A(t)\in\mathcal{M}_1} \rho(\Phi(t, t_0)). \tag{41}$$

Then for (40) *we have*

$$\begin{aligned} \Re(\mathcal{M}_1) &= 0 \quad \textit{for} \quad \delta < \hat{\delta} \\ \Re(\mathcal{M}_1) &= 1 \quad \textit{for} \quad \delta = \hat{\delta} \\ \Re(\mathcal{M}_1) &= \infty \quad \textit{for} \quad \delta > \hat{\delta}. \end{aligned}$$

It is not known how to compute $\hat{\delta}$ in general, but numerical procedures can bound $\hat{\delta}$ from above and below.

Remark 7. *A lower bound* $\underline{\delta}$ *for* $\hat{\delta}$ *can be improved by finding the largest* δ *for which there exists an* $i \geq 1$ *and a* P_i *such that Theorem 2 is satisfied for* (40).

Proposition 8 suggests a simple numerical procedure to bound $\hat{\delta}$ from above by constructing worst-case trajectories and studying the eigenvalues of the resulting discrete transition matrix.

Proposition 9. *If there exists a finite sequence* (t_k) *for some* $k = m, m+1, \ldots, L$, *where* $m \geq 0$, *such that* $A_w(t_k; V)$ *is given by* (39) *for some* $V(x)$ *along a trajectory of* (40), *and if*

$$A_d = e^{A_w(t_{L-1};V)(t_L - t_{L-1})} \ldots e^{A_w(t_0;V)(t_{m+1} - t_m)} \tag{42}$$

has an eigenvalue with magnitude 1 (and the rest less than or equal to 1 in magnitude), then $\delta = \overline{\delta} \geq \hat{\delta}$.

A simple numerical procedure can search over trajectories of (40) to produce an A_d which satisfies Proposition 9 within some tolerance. We leverage the Lyapunov function hierarchy since the tightest bound on the stability margin of (40) is achieved by using a Lyapunov function for (40) of the highest order that is computationally tractable.

1) Find the largest level i for which $\underline{\delta}$ can be computed via the procedure suggested in Remark 7 when (40) is stable.

2) Increase δ so that $\delta = \underline{\delta} + \varepsilon$. Simulate a trajectory over a discrete sequence of times (t_n) with $x_{n+1} = e^{A_w(t_n;V)(t_{n+1}-t_n)}x_n$, where $A_w(t_n;V)$ is computed at each time step t_n by solving (39) using x_n. Then see if there is a subsequence (t_k) of (t_n) such that an A_d given by (42) can be found that satisfies Proposition (9). The subsequence that maximizes the spectrum of A_d will appear by selecting k to be the times when $A_w(t_k;V) \neq A_w(t_{k-1};V)$, *i.e.* the times when $A(t)$ “switches."
3) If the eigenvalues of A_d are all less than 1 in magnitude, increase δ and test the trajectory again. If computationally tractable, increasing i further for the stable system will result in a smaller value of $\overline{\delta}$.

In practice, the system (40) often transitions from stability to instability through a limit cycle. In other words, for $A(t) \in \{A,\ A + \delta A_0\}$ where A and $A + \delta A_0$ are Hurwitz, there exists a function $\Delta(t)$ such that, for $\delta = \hat{\delta}$, (40) can produce a periodic (but not diverging) trajectory. This is not proven, however; a trajectory for a system with $\Re(\mathcal{M}_1) = 1$ may simply evolve non-periodically about a level set of a positive convex function. Though not an LTV system, a trajectory of the Lorenz attractor displays this type of behavior [35].

C. Examples

Example 5 (Aircraft Lateral Dynamics)**.** *Consider the linearized, non-dimensional lateral dynamics of a fixed-wing aircraft with the following parameters*

$$A = \begin{bmatrix} -3.088 & 0 & -1425.042 & 4.5956 \\ -18.906 & -166.878 & 29.223 & 0 \\ 6.762 & 4.445 & -19.389 & 0 \\ 0 & 1428.6 & 0 & 0 \end{bmatrix} \text{ from [36, pp. 188, 361]}, \quad A_0 = \begin{bmatrix} -1 & 0 & -10 & 10 \\ -10 & -10 & 10 & 0 \\ 10 & 10 & -10 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix} \tag{43}$$

and with non-dimensional state vector $x = [\beta\ p\ r\ \phi]^T$ representing the sideslip angle, roll rate, yaw rate, and roll angle, respectively. The nonlinear model is linearized around the equilibrium when the jet airplane is cruising at 40,000 ft. and 0.8 Mach number, so the linearized model is valid in the flight regime close to the equilibrium. The matrix A_0 represents a notional perturbation, possibly time-varying, of the vehicle's stability derivatives and mass. The time variable t has also been non-dimensionalized in this representation, and the dynamics have been scaled in order to aid in visualization. Applying the hierarchy (17) *and using the procedure suggested in Remark 7 with $i = 3$ to produce a 6^{th} order Lyapunov function, we find $\underline{\delta} = 0.24$. Applying Proposition 9 using this Lyapunov function, a limit cycle is found along the worst-case trajectory produced by* (39) *when $\overline{\delta} = 0.27$ for $(t_k) = \{0,\ 0.027,\ 0.060\}$. In other words, a*

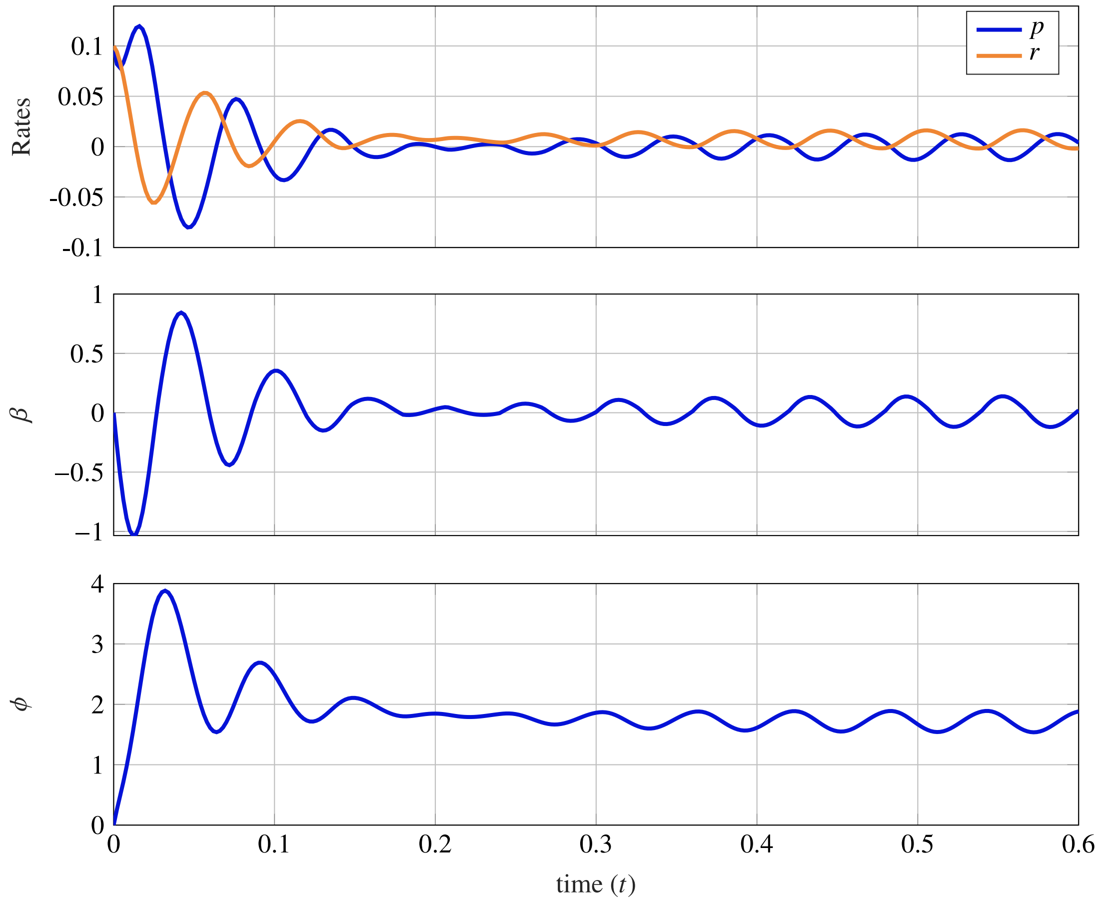

Fig. 6 Non-dimensional (including time) sample trajectory of aircraft state using $A_w(t;V)$. A limit cycle begins at about $t = 0.3$.

trajectory which evolves according to

$$A_w(t;V) = \begin{cases} A + \overline{\delta} A_0, & \textit{for} \;\; 0 \leq t < 0.027 \\ A & \textit{for} \;\; 0.027 \leq t < 0.060 \end{cases} \tag{44}$$

is a worst-case trajectory and is approximately marginally stable as certified by constructing the discrete transition matrix (42)*, which has four eigenvalues which are approximately 0, 1, and* $0.5 \pm 0.024i$*. Therefore, we know that* $0.24 < \hat{\delta} < 0.27$*. The periodic trajectory that results from using* $A_w(t;V)$ *seems to combine an undamped spiral mode with a Dutch Roll-like behavior; a sample trajectory is shown in Figure 6.*

It may be possible to bound $\hat{\delta}$ *more tightly, but increasing i to attain a higher-order Lyapunov function would make computation intractable without additionally employing an algorithm to reduce the redundancies produced by the use of the Kronecker product. This is an item of future work.*

Example 6 (Worst-Case Impulse Response)**.** *Consider the system* (23) *with the parameters*

$$A = \begin{bmatrix} 0 & 1 \\ -0.15 & -0.8 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{45}$$

and $A(t) \in \{A, A + A_0\}$. *We study the impulse response of the system by constructing a worst-case trajectory using Proposition 7 and an exponential bound using Theorem 5. Results using both* $i = 1$ *and* $i = 7$ *are compared. First, we use Proposition 7 with* $i = 7$ *to generate* $A_w(t; V_7)$ *and produce a worst-case impulse response. We also use Theorem 5 with* $i = 7$ *and* $\alpha = 0.1$ *to compute* $\bar{h}_7$*, which we use to construct an exponential bound on the impulse response. For comparison, we do the same with* $i = 1$*, using Proposition 7 to construct* $A_w(t; V_1)$ *and Theorem 5 with* $\alpha = 0.042$ *to compute* $\bar{h}_1$*. For both* $i = 7$ *and* $i = 1$*, an exponential bound is computed using the largest possible* α*. Figure 7 shows how the two worst-case impulse responses push the system towards the exponential bound. Clearly, the analysis with* $i = 7$ *produces a tighter exponential bound and a trajectory that can approach the bound more closely.*

VI. Application to Maneuver Analysis of a Microgravity Vehicle

Aerospace vehicles can be used to produce microgravity environments via atmospheric flight. Most famously, the Vomit Comet flies parabolic trajectories to provide about thirty seconds of weightlessness to passengers. The Vomit Comet is flown by hand, and automatic control can produce much more accurate microgravity environments. Recently, a prototype quadcopter was built in order to validate this possibility. This section briefly summarizes the stability analysis of the quadcopter's microgravity-tracking maneuver and shows how using the system hierarchy (17) can improve the stability analysis based on quartic Lyapunov functions for the system. The original work in [37] can be consulted for additional details about the controller design and analysis. The vehicle and its closed-loop control architecture are shown in Figure 8.

Given the controller $C(s) = c_c^T(sI - A_c)b_c$, propulsion system $G_p(s) = c_p^T(sI - A_p)b_p$, and accelerometer dynamics $G_a(s) = c_a^T(sI - A_a)b_a$, the closed-loop dynamics of the microgravity vehicle are

$$\dot{v} = c_p^T x_p - \rho v^2 + g \tag{46}$$

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A_p & b_p c_c^T & -b_p d_c c_a^T \\ 0 & A_c & -b_c c_a^T \\ b_a c_p^T & 0 & A_a \end{bmatrix} \begin{bmatrix} x_p \\ x_c \\ x_a \end{bmatrix} + \begin{bmatrix} b_p d_c \\ b_c \\ 0 \end{bmatrix} (a_d - g) + \begin{bmatrix} 0 \\ 0 \\ -b_a \rho v^2 \end{bmatrix} \tag{47}$$

where the various linear subsystems of the form $G(s) = c^T(sI - A)b$ are expressed by their state space realization and where ρ is a drag coefficient. The controller is designed to track a trajectory $v(t) = a_d t$, where a_d is a constant that

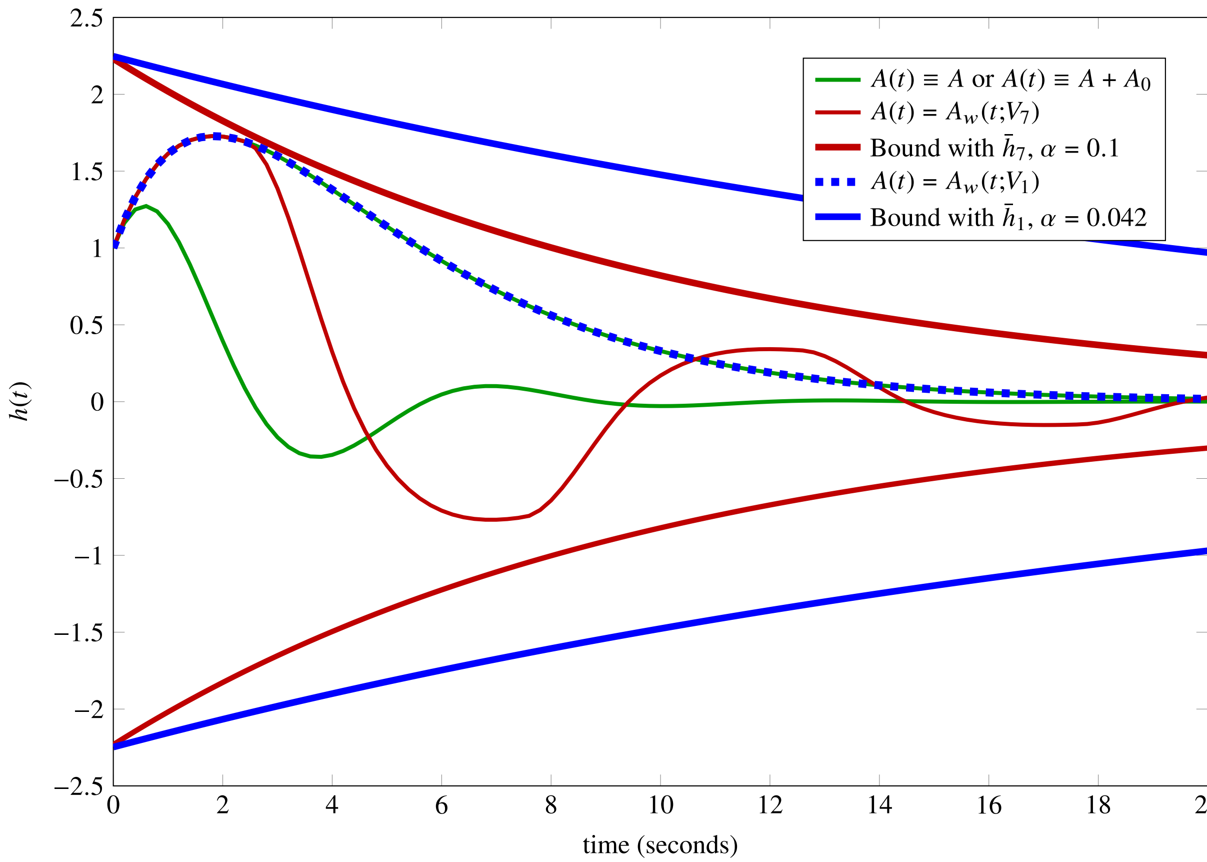

Fig. 7 Example 6. Impulse response plots show how using $A_w(t;V)$ can cause the impulse response to approach an exponential bound and how the trajectory can approach the bound more closely using a larger value of i. For the case where a quadratic Lyapunov function is used to construct $A_w(t;V_1)$, the trajectory is not time-varying and in fact lies on top of the impulse response when $A(t) \equiv A$. The best possible exponential bound produced using $\bar{h}_1$ from a quadratic Lyapunov function is shown; it is clearly not as tight as the best possible bound produced using $i = 7$, which allows us to use a larger α.

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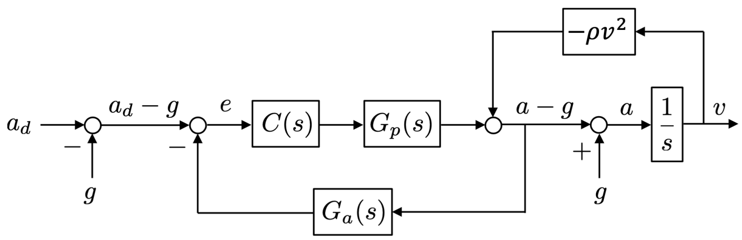
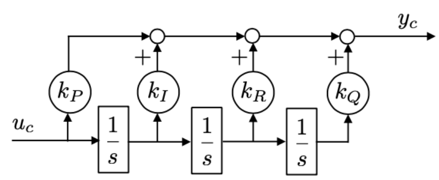
(a) Autonomous, reduced-g research vehicle during an experimental flight test.



(b) Control architecture.



(c) Controller $C(s)$.

Fig. 8 Microgravity quadcopter produces a low- or zero-gravity environment onboard [37].

specifies the desired acceleration that will produce the target microgravity environment. Clearly, the dynamics of $v(t)$ are nonlinear. By selecting a new independent variable and coordinate system, an equivalent linear system can be derived. The concept of "maneuver regulation," discussed extensively in [38], is used to perform this analysis, and the derivation of the maneuver-regulating controller is discussed in detail in [37].

An affine transformation of x_p, x_c, and x_a, parameterized by v, gives a state vector $z = [z_p \; z_c \; z_a]^T \in \mathbb{R}^n$ that evolves according to the LTV system

$$\frac{dz}{d\theta} = A(\theta)z \tag{48}$$

where the "time" variable evolves according to $\dot{\theta} = 1 + (1/a_d)c_p^T z_p$, $\theta = v/a_d$. Certifying that the maneuver is exponentially attractive amounts to proving that $z \to 0$ exponentially. To show exponential stability, the condition for quadratic stability (8) is modified to $A_j^T P + PA_j + Q \preceq 0$ for some positive-definite $Q \in S_{++}^n$. In [37], a quadratic Lyapunov function certifies the stability of the maneuver for a range of $v = 0$ to 280 m/s when $a_d = 9.807$ m/s^2. Lifting (48) with the hierarchy (17) improves the analysis. By constructing H_2 from (17) and using Theorem 2 to search for a 4-th order homogeneous polynomial Lyapunov function in z, the maneuver stability is certified for $v = 0$ to 391 m/s. Of course, this velocity can easily exceed the speed of sound, but the improved margin and stronger guarantee provided by analyzing H_2 is apparent. Consider the drag coefficient ρ, which is hard to identify precisely for a real system and which varies with altitude. The value used in this analysis is $\rho = 0.06775$, but uncertainty can be easily encoded by changing $\mathcal{M}_1$ to accommodate drag coefficients ranging, for example, from $0.03 \leq \rho \leq 0.09$. Certifying the maneuver over a broad range of ρ is particularly important for vehicles that traverse differences in altitude over which the density variation is appreciable. Using H_2 with this range of ρ, the maneuver is still exponentially attractive for a velocity range from $v = 0$ to 285 m/s; this certificate is not possible to produce with a quadratic Lyapunov function. In practice, $v = 285$ m/s cannot be achieved by the vehicle, suggesting that there still remains a lot of margin to certify maneuver stability for a realistic velocity range while perturbing many more system parameters in addition to ρ.

These computations were performed on a MacBook Pro with a 2 GHz Dual-Core Intel i5 processor. The system (48) has dimension $n = 7$, and a quadratic Lyapunov function was found using Matlab with CVX in under two seconds. Augmenting (48) using $i = 2$ results in a system of dimension $n = 49$, and computing P_2 for the case where ρ is constant takes 22 seconds. Computing a Lyapunov function for the case when ρ varies requires constructing $\mathcal{M}_1$ with four members instead of two; in this case, P_2 takes 26 seconds to compute.

VII. Conclusion

This paper presents an improvement of traditional stability and performance analyses of linear time-varying and switching linear systems by leveraging higher-order homogeneous polynomial Lyapunov functions, introduced as quadratic Lyapunov functions for a related hierarchy of linear time-varying systems. These improved analyses are accessible with only an elementary knowledge of state-space linear systems theory and convex programming formulations since such Lyapunov functions are merely quadratic Lyapunov functions of a system that has been lifted via a recursive Kronecker product-based procedure. As a result, optimization-based system analysis tools that compute invariant sets and peak response norms can be applied in an elegant fashion. To improve performance of the algorithms, further work can be done to reduce the dimension of the LMI feasibility problems. Indeed, taking the Kronecker product of two vectors results in a vector with redundant entries. A simple procedure for eliminating redundancies for systems with $n = 2$ is given in [31], and a similar procedure should be developed to address higher dimensional systems.

In addition, a continuous-time analog to the problem of computing the joint spectral radius of an uncertain system is proposed and addressed by using Lyapunov functions. The procedure developed enables the computation of approximate worst-case trajectories by finding a time-varying system matrix that can cause the system to become unstable. This was framed in a way that enables the computation of an upper bound on stability margin; this provides a measure that bounds the conservatism of a Lyapunov-based stability analysis. Such an analysis can have great practical benefit, as finding a counterexample or worst-case trajectory can be very time consuming using simulation-based methods.

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