Set-Based State Estimation of Mobile Robots from Coarse Range Measurements

Tony X. Lin[1], Samuel Coogan[2], Donald A. Sofge[3], and Fumin Zhang[1]

***Abstract*— This paper proposes a localization algorithm for an autonomous mobile robot equipped with binary proximity sensors that only indicate when the robot is within a fixed distance from beacons installed at known positions. Our algorithm leverages an ellipsoidal Set Membership State Estimator (SMSE) that maintains an ellipsoidal bound of the position and velocity states of the robot. The estimate incorporates knowledge of the robot's dynamics, bounds on environmental disturbances, and the binary sensor readings. The localization algorithm is motivated by an underwater scenario where accurate range or bearing measurements are often missing. We demonstrate our approach on an experimental platform using an autonomous blimp.**

***Index Terms*— Set-Membership Methods, State Estimation**

I. INTRODUCTION

Autonomous mobile robots have been employed in various operations ranging from oil spill surveys in underwater operations [1] to package deliveries with aerial robots [2]. These operations usually require accurate localization, or positional information, of the robot in order to map acquired data spatially and to guide the behavior of the robot during a mission. However, in certain scenarios, localization may be significantly more difficult. For example, underwater localization is challenging due to limited global positioning system (GPS) services and limited active perception methods (cameras or LiDAR). Instead, underwater localization methods have focused on using external acoustic beacons that can propagate through deep water [3, 4]. In most cases, these acoustic signals are used for time-of-flight ranging in order to improve dead reckoning estimates through some stochastic estimation method such as Kalman Filtering [5], Extended Kalman Filtering [6], or Particle Filtering [7].

Stochastic state estimators estimate a system's state by recursively computing the posterior density when considering known noise profiles of the system and the observations. However, such stochastic estimation methods may provide biased or inaccurate estimates when the underlying distribution assumptions are incorrect. In cases where statistical sensor descriptions are difficult to acquire, it may be preferable instead to obtain a bound on the system state. In particular, Set-Membership State Estimation (SMSE) methods assume that an unknown but bounded (UBB) disturbance acts on a system. An SMSE estimator then yields feasible sets on the state vector rather than posterior distributions, guaranteeing the state of the system lies within certain bounds if the UBB assumptions are continually met.

SMSE methods were originally investigated for the purposes of target tracking [8, 9] but have been extended to a wide variety of parameter and state estimation problems [10]. Recently, SMSE methods have also been explored for use in underwater robotics [11, 12]. Notable extensions of [11] demonstrate bounds on nonlinear system trajectories through contractor functions that are able to systematically eliminate non-feasible trajectories by evaluating sensing information forwards and backwards in time against historical state information [13]. Other work in [14, 15] has demonstrated the use of SMSE methods for range-only Simultaneous Localization and Mapping (SLAM) as well.

When range information is not available, target tracking with binary sensors has been explored in the literature in order to provide coarse estimates of the system state. Many previous works in the literature utilize the particle filter as the tracking method of choice [16, 17, 18]. Another work in [19] demonstrates an alternative method by formulating the problem as a Hidden Markov Model and computing solutions using a variant of the Viterbi algorithm. Similar to the issues in acoustic beacon ranging, each of these binary estimation methods produces a stochastic estimate of the system state and is potentially unreliable in the presence of a disturbance. The particle filter in particular may suffer in binary estimation problems as it is unclear how to properly re-sample from the prior distribution if samples violate binary constraints.

Motivated by the underwater localization scenario, in which the statistical properties of an acoustic sensor may vary due to changes in salinity and temperature [20], we propose using ellipsoidal SMSE methods to perform coarse localization of a robot when sensing information is only able to indicate local proximity to a beacon, also known as binary sensing. This scenario also arises in terrestrial or aerial localization problems, for example when the only sensing modality is Wi-Fi signal strength which may have very poor range estimation accuracy [21]. We propose an algorithm that leverages efficient LMI semi-definite programs to extract bounds on the states of the robot when only binary range measurements are available. To our knowledge, this work is the first to propose the use of ellipsoidal guaranteed state estimation methods in solving the binary sensing localization

[1] Tony X. Lin and Fumin Zhang are with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA tlin339, fumin@gatech.edu

[2] Samuel Coogan is with the School of Electrical and Computer Engineering and the School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA sam.coogan@gatech.edu

[3] Donald A. Sofge is with the Navy Center for Applied Research in Artificial Intelligence, Naval Research Laboratory, Washington D.C. 20375, USA donald.sofge@nrl.navy.mil

problem. In addition, we demonstrate that the use of SMSE methods is also able to extract bounds on the velocity of the robot despite having only coarse observations of position.

Our contributions in this work are therefore threefold: **i)** we propose an algorithm for identifying bounds on the state of a linear system using binary range measurements and exploiting efficient LMI semi-definite programs, **ii)** we refine the general approach to the problem of localization of robots using 3D binary range measurements, and **iii)** we validate our algorithm on an experimental research platform. The experimental platform is an autonomous blimp designed by the Georgia Tech Systems Research (GTSR) group [22]. Due to the lighter-than-air design of the blimp, the platform is easily subjected to aerodynamic disturbances, and serves as a good validation of the proposed method's resiliency under disturbance.

The outline of this paper is as follows. Section II formulates the SMSE problem for coarse area localization. Section III introduces the ellipsoidal SMSE algorithm used to compute the feasible sets on the system state. Section IV discusses our experimental tests using the proposed set-membership estimation method. Section V concludes the paper and describes future works.

II. PROBLEM FORMULATION

Consider a robot with motion described by the continuous-time linear dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t) \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and disturbance $d \in \mathbb{R}^n$. We assume the disturbance d is an unknown but bounded (UBB) disturbance and is contained within the ellipsoid

$$\mathscr{E}_d = \{z \in \mathbb{R}^n \mid z^T P_d z + 2z^T b_d + c_d \leq 0\} \tag{2}$$

where $P_d \in \mathbb{R}^{n\times n}$, $b_d \in \mathbb{R}^n$, and $c_d \in \mathbb{R}$ define the center, rotation, and semi-axis lengths of the ellipsoid bounding the disturbance. For this work, we focus specifically on tracking the 3D position and velocity of the autonomous vehicle so that

$$\dot{x}(t) = \begin{bmatrix} \mathbf{0} & I_3 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} x(t) + \begin{bmatrix} \mathbf{0} \\ I_3 \end{bmatrix} u(t) + d(t) \tag{3}$$

where x_1, x_2, and x_3 denote the 3D position, x_4, x_5, and x_6 denote the 3D velocity, $\mathbf{0}$ is a 3×3 block of zeros and I_3 is the 3×3 identity matrix. We sometimes write $r = (x_1, x_2, x_3)$ for the robot position. However, the results of this paper can be easily generalized to the general linear time-invariant case.

Next, we describe the beacon model used to provide sensor measurements. Given N beacons, for $j \in \{1, 2, \dots, N\}$, let $r_j \in \mathbb{R}^3$ be the position of the j^{th} beacon. Let R_j be the range within which the robot can detect the beacon indexed by j, so that at time t, the beacon detection is given by

$$Y_j(t) = \begin{cases} 1, & \text{if } \|r(t) - r_j\| \leq R_j \\ 0, & \text{if } \|r(t) - r_j\| > R_j. \end{cases} \tag{4}$$

Define ellipsoids describing the beacon range as

$$\mathscr{E}_j = \{z \in \mathbb{R}^6 \mid z^\intercal P_j z + 2z^\intercal b_j + c_j \leq 0\} \tag{5}$$

in which the parameters $P_j \in \mathbb{R}^{6\times 6}$, $b_j \in \mathbb{R}^6$, and $c_j \in \mathbb{R}$ are described as

$$P_j = \begin{bmatrix} R_j^{-2} I & 0 \\ 0 & 0 \end{bmatrix},\ b_j = \begin{bmatrix} -(R_j^{-2} r_j) \\ 0 \end{bmatrix},\ c_j = \begin{bmatrix} r_j & 0 \end{bmatrix} P_j \begin{bmatrix} r_j \\ 0 \end{bmatrix} \tag{6}$$

where $I \in \mathbb{R}^{3\times 3}$ is the identity matrix and $r_j = \begin{bmatrix} x & y & z \end{bmatrix}^\intercal$ is the j^{th} beacon's detection range. Then, equivalently, $Y_j(t) = 1$ if and only if $x(t) \in \mathscr{E}_j$. We assume here that the ellipsoids capturing the beacon ranges are sufficiently small to ensure that the beacons can only be detected from within the ellipsoid.

Remark II.1. *We note the ellipsoids defined in* (5) *are degenerate and have infinite bounds along the x_4, x_5, and x_6 directions. This reflects that the binary sensing measurements in* (4) *only produce position measurements in $\mathbb{R}^3$. We will see later that by defining the ellipsoid beacons in $\mathbb{R}^6$, we will be able to extract information about the robot's velocity as well, i.e., the estimate of the system's velocity does not grow unbounded even though direct information of velocity is never obtained from sensor measurements.*

Given the beacon sensing model and the dynamics of the system, we are interested in estimating the set of states that are consistent with the UBB disturbances and the binary sensing measurements observed during runtime. While our system operates with the continuous dynamics outlined in (3), we sample our sensors at discrete intervals Δt. As such, the problem is defined as follows.

Problem Definition. *Assume given a control input $u(t)$, an initial ellipsoid $\mathscr{E}[0] = \{z \in \mathbb{R}^6 \mid z^\intercal P[0] z + 2z^\intercal b[0] + c[0]\}$ such that the initial state vector satisfies $x(0) \in \mathscr{E}[0]$, a sampling interval Δt, beacon ellipsoids $\mathscr{E}_j$ for $j \in \{1, 2, \dots, N\}$, and measurements $Y(k\Delta t)$, construct a series of ellipsoids*

$$\mathscr{E}[k] = \{z \in \mathbb{R}^6 \mid z^\intercal P[k] z + 2z^\intercal b[k] + c[k] \leq 0\} \tag{7}$$

such that the state vector $x \in \mathbb{R}^6$ satisfies $x(k\Delta t) \in \mathscr{E}[k]$ for all $k \in \{0, 1, \dots, K\}$.

III. ELLIPSOIDAL SET-MEMBERSHIP STATE ESTIMATION

In this section we propose a solution to the guaranteed state estimation problem of constructing a series of ellipsoids that contain the state vector as described by (7). We leverage ellipsoidal reachable set approximations with ellipsoidal intersection updates in order to over-approximate the feasible sets of system (3). Our proposed algorithm computes an ellipsoidal over-approximation to the ellipsoidal intersection through LMI semi-definite programs.

A. Preliminaries on Ellipsoidal Reachable Sets

The reachable set of a system is defined as the set of all states that an input is capable of driving a system to given an initial set of states and a finite time horizon [23]. Many approaches have been leveraged in the literature to efficiently over-approximate reachable sets [24, 23], as exact computation of these sets is prohibitively expensive. In this work, we use ellipsoidal over-approximations to characterize the feasible set of states that satisfies the beacon sensing measurements and the disturbance inputs. To this end, we now propose a recursive algorithm for computing $\mathscr{E}[k]$ from $\mathscr{E}[k-1]$. Let some set containing the state at sampling time $(k-1)\Delta t$ be defined as the ellipsoid

$$\mathscr{E}[k-1] = \{z \in \mathbb{R}^6 \mid z^\intercal P[k-1]z + 2z^\intercal b[k-1] + c[k-1] \leq 0\} \quad (8)$$

and the bound on the total disturbance on the system be described by the ellipsoid in (2). The reachable set $R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t))$ over-approximates bounds on the state $x(k\Delta t)$ based only on system dynamics and disturbance, i.e., before incorporating any sensor measurements. For systems of the form (1), the computation of the reachable set is

$$R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t)) = \Phi(k\Delta t, (k-1)\Delta t)\mathscr{E}[k-1] \oplus \int_{(k-1)\Delta t}^{k\Delta t} \Phi(k\Delta t, \tau)(Bu(\tau) + \mathscr{E}_d)d\tau \quad (9)$$

where $\Phi(t_1, t_0) = e^{A(t_1 - t_0)}$ is the usual state transition matrix, $\oplus$ is the geometric (Minkowski) sum operator [25]. For this work, we leverage an open-source Matlab toolbox to perform the Minkowski sums of ellipsoids computation as in (9) [23].

Using (9), one solution to the guaranteed state estimation problem is choosing $\mathscr{E}[k] = R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t))$. However, the ellipsoid produced by this reachable set grows with each sampling iteration because it does not include any sensor measurements. Using the measurement model (4), we next describe how binary sensing measurements can be incorporated at each sampling time to constrain the ellipsoid estimates.

B. Ellipsoidal Bounding Intersections

Define the ellipsoidal approximation of the reachable set computed by (9) as $\mathscr{E}[k]^- = R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t))$ in which $\mathscr{E}[k]^-$ is of the form

$$\mathscr{E}[k]^- = \{z \in \mathbb{R}^6 \mid z^\intercal P[k]^- z + 2z^\intercal b[k]^- + c[k]^- \leq 0\}. \quad (10)$$

for appropriate $P[k]$, $b[k]$, and $c[k]$ as computed above. Define $\mathscr{M}[k] = \{j \mid Y_j(k\Delta t) = 1\} \subseteq \{1, 2, \ldots, N\}$ be the set of indices at sampling time $k\Delta t$ such that $j \in \mathscr{M}[k]$ if and only if $Y_j(k\Delta t) = 1$. The intersection of all ellipsoids $\mathscr{E}_j$ where $j \in \mathscr{M}[k]$ and the reachable set $\mathscr{E}[k]^-$ provides a bound reduction on the feasible states of state vector $x[k\Delta t]$. However, since the intersection is not an ellipsoid, we propose choosing the next ellipsoid $\mathscr{E}[k]$ such that

$$\mathscr{E}[k] \supset \mathscr{E}[k]^- \cap \bigcap_{j \in \mathscr{M}[k]} \mathscr{E}_j. \quad (11)$$

Fig. 1 shows a visual representation of the intersection performed in (11) that updates $\mathscr{E}[k]$ according to the intersections of beacon ellipsoids $\mathscr{E}_j$ and the ellipsoidal approximation of the reachable set $\mathscr{E}[k]^-$. We now propose a semi-definite program for computing $\mathscr{E}[k]$ satisfying (11), constituting one of the main contributions of this work.

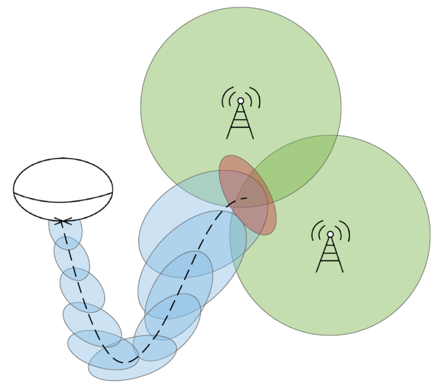

Fig. 1: Visual representation of the operation in (11) projected into 2D space. The GT-MAB is depicted computing bounds on its state using ellipsoidal reachable set approximation (in blue). Upon detecting beacons (with ranges shown in green) the ellipsoidal approximation of the reachable set $\mathscr{E}[k]^-$, is updated with the intersection of the ellipsoid regions to produce ellipsoid estimate $\mathscr{E}[k]$ (in red).

Proposition III.1. *Given $\mathscr{M}[k]$ such that $j \in \mathscr{M}[k]$ if and only if $Y_j(k\Delta t) = 1$. If $P[k]$, $b[k]$, $c[k]$, τ_k, and $\{\tau_j\}_{j \in \mathscr{M}[k]}$ satisfy the LMI*

$$\begin{bmatrix} P[k] & b[k] & 0 \\ b[k]^\intercal & -1 & b[k]^\intercal \\ 0 & b[k] & -P[k] \end{bmatrix} - \tau_k \begin{bmatrix} P[k]^- & b[k]^- & 0 \\ (b[k]^-)^\intercal & c[k]^- & 0 \\ 0 & 0 & 0 \end{bmatrix} - \sum_{j \in \mathscr{M}[k]} \tau_j \begin{bmatrix} P_j & b_j & 0 \\ b_j^\intercal & c_j & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0, \quad (12)$$

then $\mathscr{E}[k]$ satisfies (11).

Proof. Let $T(z; P, b, c) = z^\intercal P z + 2z^\intercal b + c$ describe an ellipsoid such that $\mathscr{E} = \{z \mid T(z; P, b, c) \leq 0\}$. Condition (11) is true if and only if $T(z; P[k], b[k], c[k]) \leq 0$ for any z such that $T(z; P[k]^-, b[k]^-, c[k]^-) \leq 0$ and $T(z; P_j, b_j, c_j) \leq 0$ for all $j \in \mathscr{M}[k]$ where $\mathscr{E}[k]^- = \{z \in \mathbb{R}^6 \mid T(z; P[k]^-, b[k]^-, c[k]^-) \leq 0\}$ and $\mathscr{E}_j = \{z \in \mathbb{R}^6 \mid T(z; P_j, b_j, c_j) \leq 0\}$. By the S-procedure [26], this holds if there exist positive scalars $\tau_k, \tau_j, j \in \mathscr{M}[k]$ such that

$$T(z; P[k], b[k], c[k]) - \tau_k T(z; P[k]^-, b[k]^-, c[k]^-) - \sum_{j \in \mathscr{M}[k]} \tau_j T(z; P_j, b_j, c_j) \leq 0, \quad \forall z \in \mathbb{R}^6 \quad (13)$$

which is equivalently written as

$$\begin{bmatrix} P[k] & b[k] \\ b^\intercal[k] & b[k]^\intercal P[k]^{-1} b[k] - 1 \end{bmatrix} - \tau_k \begin{bmatrix} P[k]^- & b[k]^- \\ (b[k]^-)^\intercal & c[k]^- \end{bmatrix} - \sum_{j \in \mathscr{M}[k]} \tau_j \begin{bmatrix} P_j & b_j \\ b_j^\intercal & c_j \end{bmatrix} \leq 0. \quad (14)$$

By considering the Schur complement of (12), (14) is shown to be equivalent to (12). □

Proposition III.1 allows for constructing an LMI with $P[k]$ and $b[k]$ as decision variables such that the resulting ellipsoid

$$\mathscr{E}[k] = \{z \in \mathbb{R}^6 \mid z^\intercal P[k]z + 2z^\intercal b[k] + c[k] \leq 0\}. \quad (15)$$

where $c[k] = b[k]^T P[k]^{-1} b[k] - 1$ satisfies (11), i.e., solves the guaranteed state estimation problem. This algorithm is formalized next.

Algorithm 1: Proximity Sensing Ellipsoidal SMSE

Input : P_0, b_0, c_0, $\mathscr{E}_u$, K
Output: $\mathscr{E} = \{\mathscr{E}_k : k \in \{0, 1, \ldots, K\}\}$

k = 1
$\mathscr{E}_{k-1} \leftarrow T(P_0, b_0, c_0)$
$\mathscr{E}[0] = T(P_0, b_0, c_0)$
while $k \leq K$ **do**
  $\mathscr{M}_k \leftarrow \{j \in \{1, \ldots, N\} : Y_j(k\Delta t) = 1\}$
  $P_k^-, b_k^- \leftarrow R(k\Delta t, (k-1)\Delta t, \mathscr{E}_{k-1}, \mathscr{E}_d, u(t))$ ▷ as in (9)
  $c_k^- \leftarrow (b_k^-)^\intercal (P_k^-)^{-1} b_k^- - 1$
  $\beta_k^- \leftarrow \begin{bmatrix} P_k^- & b_k^- & 0 \\ (b_k^-)^\intercal & c_k^- & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  $\beta_j \leftarrow \begin{bmatrix} P_j & b_j & 0 \\ b_j^\intercal & c_j & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \forall j \in \mathscr{M}_k$
  `Solve`(P_k, b_k) ▷ as in (12)
    minimize $\log(\det(P_k^{-1}))$
    subject to $P_k \geq 0$,
    $\tau_k \geq 0, \tau_j \geq 0, j \in \mathscr{M}_k$,
    $\beta_k - \tau_k \beta_k^- - \sum_{j \in \mathscr{M}_k} \tau_j \beta_j \leq 0$
  $k += 1$
  $c_k \leftarrow b_k^\intercal P_k^{-1} b_k - 1$
  $\mathscr{E}[k] = T(P_k, b_k, c_k)$
  $\mathscr{E}_{k-1} \leftarrow \mathscr{E}[k]$
end

C. Ellipsoidal SMSE

The Proximity Sensing Ellipsoidal SMSE algorithm combines reachability and ellipsoidal intersection to maintain a guaranteed bound that contains the state of the system at all time when considering the measurement model in (4). This characteristic can be easily seen from the definitions of reachability and ellipsoidal intersection. The full ellipsoidal SMSE algorithm is shown in Algorithm 1.

Theorem III.1. *The set $\mathscr{E}[k]$ computed by the Proximity Sensing Ellipsoidal SMSE algorithm is guaranteed to contain the state $x(k\Delta t)$ if the initial state $x(0)$ is contained within the initial set $\mathscr{E}[0]$ and the disturbances applied to the system are contained within $\mathscr{E}_d$.*

Proof. In view of the reachable set approximation (9) and the LMI (12) derived in Proposition III.1, $R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t))$ bounds the state of the system and $\mathscr{E}[k]$ bounds the intersection of $R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t))$ with $\mathscr{E}_j$, for all detected beacons $j \in \mathscr{M}[k]$. As such, for the detection signal $Y(k\Delta t)$ given $x(0) \in \mathscr{E}[0]$, $R(k\Delta t, (k-1)\Delta t, \mathscr{E}[k-1], \mathscr{E}_d, u(t)) \cap \mathscr{E}_j$, $\forall j \in \mathscr{M}[k]$ bounds the feasible set of states such that $x(k\Delta t) \in \mathscr{E}[k]$ for all $k \geq 0$. □

Fig. 2: Flight trajectory of the GT-MAB during the experiment in 3D space. The blimp on the left is the starting point of the sequence and the blimp on the right is the ending point of the sequence.

Note that the ellipsoidal bound is obtained for the state of the robot, which contains both the position and the velocity. Hence our method is effective even though only binary measurements for the relative positions between the robot and the beacons are used. By including the beacon measurements through the LMI in (12), bounds on the velocity will not grow to infinity, as is the case if only the ellipsoidal reachable set approximation computation in (9) is used.

IV. EXPERIMENTS

We evaluate the proposed estimator on an experimental testbed of an autonomous system. In this section, we use our estimator to extract position and velocity bounds on the Georgia Tech Miniature Autonomous Blimp (GT-MAB) [22] while it completes a mission. The GT-MAB is a custom research platform consisting of lighter-than-air micro blimps that utilize five propellers to move freely in the 3D space.

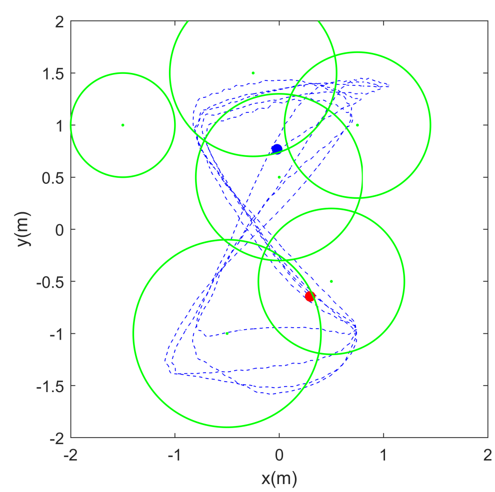

Fig. 3: Projected x-y trajectory of the GT-MAB in the designated workspace. Virtual beacon ranges are plotted as green circles and the GT-MAB trajectory is shown in dotted blue. Starting and ending positions of the GT-MAB are given by the blue and red dots.

The blimp dynamics are as in (3) when the heading has been fixed. When considering the underwater scenario, the GT-MAB can be used to model an underwater vehicle's motion as the aerodynamic effects from an environment (e.g.

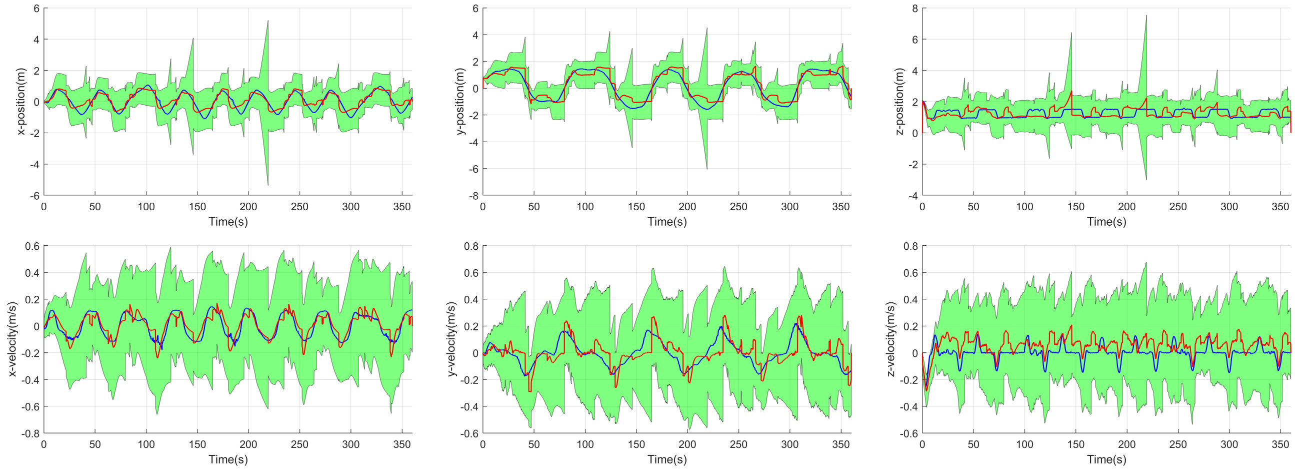

Fig. 4: Ellipsoidal set-membership computed bounds on $x(t)$ using the GT-MAB. True state is shown in blue, ellipsoid center is shown in red, and error bounds are shaded in green. The experiment was run for 6 minutes.

indoor air conditioning, aerodynamic drag) constitute disturbances that are similar to the hydrodynamic disturbances experienced underwater. Fig. 2 depicts a GT-MAB platform flying a 3D trajectory during the experiment, illustrating the motion of the blimp as it moves back towards the whiteboard from left to right. In these experiments, the disturbance ellipsoid $\mathscr{E}_d$ is a ball with radius $r = 0.01$ which was experimentally found by applying a sequence of inputs to the blimp and comparing the measured trajectories with their expected trajectories according to (3). We found that a ball with radius $r = 0.01m$ was sufficient to capture the natural aerodynamic disturbances arising from the air conditioning in the room. We use a sampling rate of $\Delta t = 0.1s$ for our experiments. Leveraging Algorithm 1, we demonstrate that we are able to compute bounds on the state of the GT-MAB throughout the duration of the experiment.

Remark IV.1. *We note that while the beacon ellipsoids $\mathscr{E}_j$ are degenerate as noted in Remark II.1 and only contain position information, the constraint in* (12) *allows for a semi-definite program to obtain velocity bounds that are implied by the position measurements, i.e. measurements of only position nonetheless provide information about velocity.*

In this experiment, the GT-MAB visits the waypoints $(1,1,1)$, $(-1,1,1.5)$, $(1,-1,1)$ and $(-1,-1,1.5)$ defined with respect to the center of the room. The GT-MAB holds a constant heading and tracks the waypoints sequentially, switching to the next waypoint once the blimp is within $0.25m$. Waypoint and heading tracking is performed using the Autopilot software as detailed in [27]. The beacon network is simulated using an Optitrack motion capture system according to the measurement model in (4). In total, six beacons are used to facilitate the set-membership estimation with ranges and positions listed in Table I. The beacons are placed in an irregular configuration with non-uniform ranges in order to test the versatility of the estimator given an arbitrary beacon network. Fig. 3 depicts the six beacon network's configuration and the blimp's hourglass-like path denoted by the dotted blue curve starting at the blue dot and ending at the red dot. As shown in Fig. 4, the estimator is able to extract accurate bounds on the true position and velocity of the blimp in 3D space and the center of the estimate tracks the true state closely. We note that the position bounds grow very large at time $t \approx 140s$ and $t \approx 220s$. This is due to the blimp briefly leaving all beacon detection regions. However as the blimp returns to a beacon region, we can see the estimated bound once again tightly encloses the state.

Beacon ID	1	2	3	4	5	6
X	0m	-0.5m	-1.5m	0.75m	0.5m	-0.25m
Y	0.5m	-1m	1m	1m	-0.5m	1.5m
Z	1.5m	1m	1.5m	1m	1.5m	1m
Range	0.8m	0.9m	0.5m	0.7m	0.7m	0.8m

TABLE I: Beacon positions and ranges for the live GT-MAB experiment.

Our Matlab implementation computes each iteration (reachable set estimation and intersection approximation) in $\approx 1s$. We note that, in this experimental setup, the control input $u(t)$ is not computed using the bounds produced by the proposed SMSE algorithm, i.e., the state-estimate is not used in feedback.

V. CONCLUSIONS

In this paper, we presented a solution to the proximity sensing localization problem capable of extracting guaranteed bounds on the position and velocity of a robot. This problem is commonly solved using stochastic estimation methods such as the Particle Filter or Kalman Filter, however these methods may yield biased estimates when disturbances are present. We demonstrated instead that the computation of reachable sets and ellipsoidal intersections are able to track the true position and velocity and guarantee error bounds on the tracking estimate even when subjected to unknown but bounded disturbances. In addition, we showed that the estimator is able to extract velocity bounds from the coarse position observations by using degenerate ellipsoids to induce bound restrictions based on the dynamics of the system. In future works, we are interested in studying the tracking performance of a controller that relies on the state estimate computed by the Proximity Sensing Ellipsoidal

SMSE algorithm, i.e., uses the ellipsoidal bounds as feedback to compute $u(t)$. By considering bounds on the state estimate, we aim to derive tight bounds on the tracking error.

ACKNOWLEDGEMENTS

F. Zhang and T. Lin were partially supported by supported by ONR grants N00014-19-1-2556 and N00014-19-1-2266; AFOSR grant FA9550-19-1-0283; NSF grants CNS-1828678, S&AS-1849228 and GCR-1934836; NRL grants N00173-17-1-G001 and N00173-19-P-1412 ; and NOAA grant NA16NOS0120028. S. Coogan was partially supported by NSF grant 1836932.

REFERENCES

[1] Shayok Mukhopadhyay et al. "Collaborative autonomous surveys in marine environments affected by oil spills". In: *Cooperative Robots and Sensor Networks 2014*. Springer, 2014, pp. 87–113.

[2] Hang Ma et al. "Multi-agent path finding with payload transfers and the package-exchange robot-routing problem". In: *Thirtieth AAAI Conference on Artificial Intelligence*. 2016.

[3] Paul Newman and John Leonard. "Pure range-only sub-sea SLAM". In: *Robotics and Automation, 2003. Proceedings. ICRA'03. IEEE International Conference on*. Vol. 2. Ieee. 2003, pp. 1921–1926.

[4] Edwin Olson, John J Leonard, and Seth Teller. "Robust range-only beacon localization". In: *IEEE Journal of Oceanic Engineering* 31.4 (2006), pp. 949–958.

[5] Rudolph Emil Kalman. "A new approach to linear filtering and prediction problems". In: *Journal of basic Engineering* 82.1 (1960), pp. 35–45.

[6] Andrew H Jazwinski. *Stochastic processes and filtering theory*. Courier Corporation, 2007.

[7] Dieter Fox et al. "Particle filters for mobile robot localization". In: *Sequential Monte Carlo methods in practice*. Springer, 2001, pp. 401–428.

[8] Fred Schweppe. "Recursive state estimation: Unknown but bounded errors and system inputs". In: *IEEE Transactions on Automatic Control* 13.1 (1968), pp. 22–28.

[9] D Bertsekas and I Rhodes. "Recursive state estimation for a set-membership description of uncertainty". In: *IEEE Transactions on Automatic Control* 16.2 (1971), pp. 117–128.

[10] Boris T Polyak et al. "Ellipsoidal parameter or state estimation under model uncertainty". In: *Automatica* 40.7 (2004), pp. 1171–1179.

[11] Luc Jaulin. "Robust set-membership state estimation; application to underwater robotics". In: *Automatica* 45.1 (2009), pp. 202–206.

[12] Renata Neuland et al. "Hybridization of Monte Carlo and set-membership methods for the global localization of underwater robots". In: *2014 IEEE/RSJ International Conference on Intelligent Robots and Systems*. IEEE. 2014, pp. 199–204.

[13] Simon Rohou et al. "Reliable non-linear state estimation involving time uncertainties". In: *Automatica* 93 (2018), pp. 379–388.

[14] Luc Jaulin. "Range-only slam with occupancy maps: A set-membership approach". In: *IEEE Transactions on Robotics* 27.5 (2011), pp. 1004–1010.

[15] Wen Yu, Erik Zamora, and Alberto Soria. "Ellipsoid SLAM: a novel set membership method for simultaneous localization and mapping". In: *Autonomous Robots* 40.1 (2016), pp. 125–137.

[16] Sungjin Cho and Fumin Zhang. "Localization of Autonomous Underwater Vehicles Incorporating Flow Models and Acoustic Detection". In: *Proceedings of the 10th International Conference on Underwater Networks & Systems*. ACM. 2015, p. 34.

[17] Jaspreet Singh et al. "Tracking multiple targets using binary proximity sensors". In: *Proceedings of the 6th international conference on Information processing in sensor networks*. ACM. 2007, pp. 529–538.

[18] Petar M Djuric, Mahesh Vemula, and Mónica F Bugallo. "Target tracking by particle filtering in binary sensor networks". In: *IEEE Transactions on Signal Processing* 56.6 (2008), pp. 2229–2238.

[19] Songhwai Oh and Shankar Sastry. "Tracking on a graph". In: *Proceedings of the 4th international symposium on Information processing in sensor networks*. IEEE Press. 2005, p. 26.

[20] Sandra Sendra et al. "Underwater acoustic modems". In: *IEEE Sensors Journal* 16.11 (2016), pp. 4063–4071.

[21] Chenshu Wu et al. "Mitigating large errors in WiFi-based indoor localization for smartphones". In: *IEEE Transactions on Vehicular Technology* 66.7 (2016), pp. 6246–6257.

[22] Ningshi Yao et al. "Monocular vision-based human following on miniature robotic blimp". In: *Robotics and Automation (ICRA), 2017 IEEE International Conference on*. IEEE. 2017, pp. 3244–3249.

[23] Alex A Kurzhanskiy and Pravin Varaiya. "Ellipsoidal toolbox (ET)". In: *Decision and Control, 2006 45th IEEE Conference on*. IEEE. 2006, pp. 1498–1503.

[24] Antoine Girard. "Reachability of uncertain linear systems using zonotopes". In: *International Workshop on Hybrid Systems: Computation and Control*. Springer. 2005, pp. 291–305.

[25] Steven S Skiena. *The algorithm design manual: Text*. Vol. 1. Springer Science & Business Media, 1998.

[26] Stephen Boyd et al. *Linear matrix inequalities in system and control theory*. Vol. 15. Siam, 1994.

[27] Sungjin Cho et al. "Autopilot design for a class of miniature autonomous blimps". In: *2017 IEEE Conference on Control Technology and Applications (CCTA)*. IEEE. 2017, pp. 841–846.