Approximating Periodic Trajectories of Contractive Systems

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***Abstract*— We consider contractive systems whose trajectories evolve on a compact and convex state-space. It is well-known that if the time-varying vector field of the system is periodic then the system admits a unique globally asymptotically stable periodic solution. Obtaining explicit information on this periodic solution and its dependence on various parameters is important both theoretically and in numerous applications. We develop an approach for approximating such a periodic trajectory using the periodic trajectory of a simpler system (e.g. an LTI system). Our approximation includes an error bound that is based on the input-to-state stability property of contractive systems. We show that in some cases this error bound can be computed explicitly. We demonstrate our results using several examples from systems biology.**

I. Introduction

A dynamical system is called *contractive* if any two trajectories approach each other [1], [2]. This is a strong property with many important implications. For example, if the trajectories evolve on a compact and convex state-space Ω then the system admits an equilibrium point $e \in \Omega$, and since every trajectory converges to the trajectory emanating from e, e is globally asymptotically stable. Note that establishing this does not require an explicit description of e.

More generally, contractive systems with a periodic excitation *entrain*, that is, their trajectories converge to a periodic solution with the same period as the excitation. However, the proof of the entrainment property of contractive systems is based on implicit arguments (see, e.g. [3]) and provides no explicit information on the periodic trajectory (except for its period).

Contraction theory has found numerous applications in systems and control theory, systems biology [4], and more (see e.g. the recent survey [2]). A particularly interesting line of research is based on combining contraction theory and graph theory in order to study various networks of multi-agent systems (see, e.g. [5], [6], [7], [8]). Contraction theory has also been used to obtain convergence bounds for singularly perturbed systems [9], [10].

As already noted by Desoer and Haneda [11], contractive systems satisfy a special case of the input-to-state stability (ISS) property (see the survey paper [12]). Desoer and Haneda used this to derive bounds on the error between trajectories of a continuous-time contractive system and its time-discretized model. This is important when computing solutions of contractive systems using numerical integration methods [13]. Sontag [14] has shown that contractive systems satisfy a "converging-input converging output" property. A recent paper [15] used the ISS property to derive a bound on the error between trajectories of a continuous-time contractive system and those of some "simpler" continuous-time system (e.g. an LTI system). This bound is particularly useful when the simpler model can be solved explicitly.

Here, we derive new bounds on the distance between the periodic trajectory of a contractive system and the periodic trajectory of a "simpler" system, e.g. an LTI system with a periodic forcing. We show several cases where the periodic trajectory of the simpler system is explicitly known and the bound is also explicit, so this provides considerable information on the unknown periodic trajectory of the contractive system.

Due to space limitations, all the proofs of our results are omitted. See [16] for an extended version of this paper with the proofs.

II. Preliminaries

Consider the time-varying dynamical system

$$\dot{x}(t) = f(t, x(t)), \tag{1}$$

with the state x evolving on a positively invariant convex set $\Omega \subseteq \mathbb{R}^n$. We assume that $f(t,x)$ is differentiable with respect to x, and that both $f(t,x)$ and its Jacobian $J(t,x) := \frac{\partial f}{\partial x}(t,x)$ are continuous in (t,x). Let $x(t,t_0,x_0)$ denote the solution of (1) at time $t \geq t_0$ for the initial condition $x(t_0) = x_0$. For the sake of simplicity, we assume from here on that $x(t,t_0,x_0)$ exists and is unique for all $t \geq t_0 \geq 0$ and all $x_0 \in \Omega$.

The system (1) is said to be *contractive* on Ω with respect to a vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if there exists $\eta > 0$ such that

$$|x(t,t_0,a) - x(t,t_0,b)| \leq e^{-(t-t_0)\eta}|a-b| \tag{2}$$

for all $t \geq t_0 \geq 0$ and all $a, b \in \Omega$. This means that any two trajectories approach one another at an exponential rate η. This implies in particular that the initial condition is "quickly forgotten".

Note that contraction can be defined in a more general way, for example with respect to a time- and space-varying norm [1] (see also [17]). We focus here on exponential contraction with respect to a *fixed* vector norm because there exist easy to check sufficient conditions, based on matrix measures, guaranteeing that (2) holds. A vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ induces a *matrix measure* $\mu : \mathbb{R}^{n \times n} \to \mathbb{R}$ defined

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by

$$\mu(A) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(||I + \varepsilon A|| - 1),$$

where $||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is the matrix norm induced by $|\cdot|$. For example, for the ℓ_1 vector norm, denoted $|\cdot|_1$, the induced matrix norm is the maximum absolute column sum of the matrix, and the induced matrix measure is $\mu_1(A) = \max\{c_1(A), \dots, c_n(A)\}$, where $c_j(A) := A_{jj} + \sum_{i\neq j} |A_{ij}|$, i.e., the sum of the entries in column j of A, with non-diagonal elements replaced by their absolute values.

If the Jacobian of f satisfies

$$\mu(J(t,x)) \leq -\eta, \quad \text{for all } x \in \Omega \text{ and all } t \geq t_0 \geq 0, \quad (3)$$

then (2) holds (see [3] for a self-contained proof). This is in fact a particular case of using a Lyapunov-Finsler function to prove contraction [17]. We will focus on the case where $\eta > 0$, but some of our results hold when $\eta \leq 0$ as well. In this case, (2) provides a bound on how quickly can trajectories of (1) separate from one another.

Often it is useful to work with scaled vector norms (see, e.g. [18], [19]). Let $|\cdot|_* : \mathbb{R}^n \to \mathbb{R}_+$ be some vector norm, and let $\mu_* : \mathbb{R}^{n\times n} \to \mathbb{R}$ denote its induced matrix measure. If $D \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $|\cdot|_{*,D} : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm defined by $|z|_{*,D} := |Dz|_*$, then the induced matrix measure is $\mu_{*,D}(A) = \mu_*(DAD^{-1})$.

The next result describes an ISS property of contractive systems with an additive input.

Theorem 1 *[11] Consider the system*

$$\dot{x}(t) = f(t, x(t)) + u(t), \quad (4)$$

where $y \to f(t,y)$ is C^1 for all $t \geq t_0$. Fix some vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ and suppose that (3) *holds for the induced matrix measure $\mu(\cdot)$. Then the solution of* (4) *with $x(t_0) = x_0$ satisfies, for all $t \geq t_0$,*

$$|x(t,t_0,x_0)| \leq e^{-\eta(t-t_0)}|x_0| + \int_{t_0}^{t} e^{-\eta(t-s)}|u(s)|\, ds.$$

Ref. [15] has applied the ISS property to derive a bound on the error between trajectories of the contractive system (1) and those of a "simpler" dynamical system $\dot{y} = g(t, y(t))$. For such a system, pick $y_0 \in \Omega$, and let $\tau \geq t_0$ be such that the solution $y(t,t_0,y_0)$ belongs to Ω for all $t \in [t_0, \tau]$. Then the difference between the trajectories of the two systems $d(t) := x(t,t_0,x_0) - y(t,t_0,y_0)$ satisfies

$$|d(t)| \leq e^{-\eta(t-t_0)}|x_0 - y_0| + \int_{t_0}^{t} e^{-\eta(t-s)}|f(s, y(s,t_0,y_0)) - g(s, y(s,t_0,y_0))|\, \mathrm{d}s \quad (5)$$

for all $t \in [t_0, \tau]$. The proof of this result is based on noting that

$$\begin{aligned} \dot{d}(t) &= f(t,x(t)) - f(t,y(t)) + f(t,y(t)) - g(t,y(t)) \\ &= M(t)d + u(t), \end{aligned}$$

where $M(t) := \int_0^1 J(t, sx(t) + (1-s)y(t))\, \mathrm{d}s$, and $u(t) := f(t,y(t)) - g(t,y(t))$. Since $y(t) \in \Omega$ for all $t \in [0,\tau]$ and Ω is convex, $sx(t) + (1-s)y(t) \in \Omega$ for all $t \in [0,\tau]$ and all $s \in [0,1]$. Using (3) and the subadditivity of matrix measures [20], [11], which, by continuity, extends to integrals, yields $\mu(M(t)) \leq -\eta$ for all $t \in [0,\tau]$. Summarizing, $\dot{d}(t) = M(t)d(t) + u(t)$ is a contractive system with an additive "disturbance" u and applying the ISS property of contractive systems yields (5).

Note that the integrand in (5) depends on the difference between the vector fields f and g *evaluated along the trajectory of the y system*. This is useful, for example, when the trajectory of the y system is explicitly known.

The applications studied in [15] were contractive systems with time-invariant vector fields approximated by time-invariant LTI systems. Here, we consider a different case, namely, when the vector field $f(t,x)$ is time-varying and T-periodic for some $T > 0$, that is,

$$f(t,z) = f(t+T, z)$$

for all $t \geq t_0$ and all $z \in \Omega$. It is well-known that in this case every trajectory of (1) converges to a unique periodic solution $\gamma(t)$ of (1) with period T (see [3] for a self-contained proof). This entrainment property is very important in applications (see, e.g. [21], [3]). However, the proof of entrainment is based on implicit arguments and provides no information on the properties of the periodic trajectory (except for its period). Our goal here is to develop a suitable bound for the difference between $\gamma(t)$ and the periodic solution $\kappa(t)$ of some simpler approximating y system and to suggest suitable approximating systems.

III. Bounds on the difference between two periodic trajectories

In this section, we consider the T-periodic orbit of a T-periodic contractive system. Theorem 2 below is our main result in this section, and provides a bound on the distance of this periodic orbit to a T-periodic orbit of some approximating system.

Theorem 2 *Consider the system*

$$\dot{x} = f(t,x) \quad (6)$$

whose trajectories evolve on a compact and convex state-space $\Omega \subseteq \mathbb{R}^n$. Suppose that $f(t,x)$ is T-periodic and that $f(t,x)$ and $J(t,x)$ are continuous in (t,x). Let $|\cdot|$ be some vector norm on $\mathbb{R}^n$ and $\mu(\cdot)$ its induced matrix measure, and suppose that $\mu(J(t,x)) \leq -\eta < 0$ for all $t \geq 0$ and all $x \in \Omega$. Let $\gamma(t)$ be the unique periodic trajectory of (6) *with period T. Consider another time-varying system*

$$\dot{y} = g(t,y) \quad (7)$$

and suppose that $g(t,y)$ is also T-periodic and that $\kappa(t)$ is a T-periodic trajectory of (7) *with $\kappa(t) \in \Omega$ for all $t \in [0,T]$.*

Define $c : \mathbb{R}_+ \to \mathbb{R}_+$ *by*

$$c(\alpha) := \int_0^{\alpha} e^{-\eta(\alpha-s)} |f(s,\kappa(s)) - g(s,\kappa(s))| \, \mathrm{d}s. \tag{8}$$

Then the difference between the two periodic trajectories satisfies

$$|\gamma(\tau) - \kappa(\tau)| \le \frac{e^{-\eta\tau}}{1-e^{-\eta T}} c(T) + c(\tau) \tag{9}$$

for all $\tau \in [0,T]$.

Note that the bound here depends on the difference between the vector fields f and g evaluated along the periodic trajectory $\kappa(s)$ of the "simpler" y system. This is useful for example when the y system is an asymptotically stable LTI system with a sinusoidal forcing term, as then $\kappa(t)$ is known explicitly.

We now derive a simpler (and less tight) bound. By the definition of $c(\cdot)$, $c(\alpha) \le \frac{1-e^{-\eta\alpha}}{\eta} \max_{t\in[0,\alpha]} |f(t,\kappa(t)) - g(t,\kappa(t))|$ for all $\alpha \ge 0$, and combining this with (9) yields the following result.

Corollary 1 *Under the hypotheses of Theorem 2,*

$$|\gamma(\tau) - \kappa(\tau)| \le \frac{1}{\eta} \max_{t\in[0,T]} |f(t,\kappa(t)) - g(t,\kappa(t))|$$

for all $\tau \ge 0$.

This bound is useful in cases where one can establish a bound on the difference between the vector fields f and g along the periodic trajectory κ of the approximating system. Note that the bound here demonstrates a clear tradeoff: if g is "close" to f then the error $f - g$ will be small, yet κ may be an unknown, complicated trajectory (as we assume that f is a nonlinear vector field). On the other hand, if g is relatively simple (e.g., the vector field of an LTI system) then κ may be known explicitly yet the difference $|f-g|$ may be large.

To summarize, Theorem 2 and Corollary 1 provide a bound on the distance of the unique T-periodic trajectory of a contractive system and some T-periodic trajectory of an approximating system. The next step is to determine a suitable approximating system. We propose two natural approximating systems for the case where the periodic vector field arises via a periodic forcing function. The first approximating system considers the time-averaged periodic forcing function to arrive at an autonomous dynamical system with an equilibrium. The second approximating system results from a linearization of the dynamics, keeping the periodic excitation as is.

IV. Approximating Systems

From hereon, we consider a special case of the contractive system (6) with the form

$$\dot{x}(t) = f(t,x(t)) = F(x(t),u(t))$$

where $u(t)$ is a given m-dimensional, T-periodic excitation.

A. Averaging the input

Our first result is based on using a "simpler" y system derived by averaging the excitation u over a period. The excitation in the y system is thus constant. We assume that the y system admits an equilibrium point $e \in \Omega$, and apply Theorem 2 to derive a bound on the distance between the periodic trajectory $\gamma(t)$ of the original x system and the point e.

Theorem 3 *Consider the system*

$$\dot{x} = F(x,u), \tag{10}$$

where u *is an* m*-dimensional periodic excitation with period* $T \ge 0$*. Suppose that the trajectories of* (10) *evolve on a compact and convex state space* $\Omega \subset \mathbb{R}^n$*. Assume that for some vector norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ *and induced matrix measure* $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$*,*

$$\mu\left(\frac{\partial F}{\partial x}(x,u(t))\right) \le -\eta < 0$$

for all $t \ge 0$ *and all* $x \in \Omega$*. Let* $\gamma(t)$ *be the unique, attracting,* T*-periodic orbit of* (10) *in* Ω*. Then, for any* $z \in \Omega$*,*

$$|x(t,0,z) - z| \le \int_0^t e^{-\eta(t-s)} |F(z,u(s))| \, \mathrm{d}s \tag{11}$$

for all $t \ge 0$*. In particular, for all* $t \ge 0$*,*

$$|x(t,0,z) - z| \le (1-e^{-\eta t}) c/\eta,$$

where $c := \max_{t\in[0,T]} |F(z,u(t))|$*. Moreover, for all* $\tau \in [0,T]$*,*

$$\begin{aligned} |\gamma(\tau) - z| \le{}& \frac{e^{-\eta\tau}}{1-e^{-\eta T}} \int_0^T e^{-\eta(T-s)} |F(z,u(s))| \, \mathrm{d}s \\ &+ \int_0^{\tau} e^{-\eta(\tau-s)} |F(z,u(s))| \, \mathrm{d}s \qquad (12) \\ \le{}& c/\eta. \qquad (13) \end{aligned}$$

The next two examples demonstrate that a natural choice for z in Theorem 3 is the equilibrium point induced by the average of the periodic excitation.

Example 1 Our focus here is on nonlinear dynamical systems, but it is still useful to begin by considering the linear system

$$\dot{x} = Ax + Bu, \tag{14}$$

where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $B \in \mathbb{R}^{n\times m}$, and u is an m-dimensional T-periodic control. It is well-known that such a system is contractive [2], [22]. For the sake of completeness we repeat the argument here. We use the notation $Q > 0$ to denote that a matrix Q is symmetric and positive-definite. Since A is Hurwitz, there exist $\eta > 0$ and $Q > 0$ such that

$$QA + A'Q \le -2\eta Q. \tag{15}$$

Let $P > 0$ be a matrix such that $P^2 = Q$. Then multiplying (15) by P^{-1} on the left and on the right yields

$$PAP^{-1} + P^{-1}A'P \le -2\eta I. \tag{16}$$

This means that the Jacobian A of (14) satisfies $\mu_{2,P}(A) \leq -\eta$, where $\mu_{2,P}$ is the matrix measure induced by the scaled Euclidean norm $|z|_{2,P} := |Pz|_2$.Thus, (14) is contractive with respect to this scaled norm with contraction rate η, and every solution of (14) converges to the unique T-periodic solution $\gamma(t)$ of (14). Let $\bar{u} := \frac{1}{T}\int_0^T u(s)ds$ and choose $z = A^{-1}B\bar{u} =: e$, the equilibrium of the time-invariant system with input equal to $\bar{u}$.

To apply the bound (13), note that $F(e, u(s)) = Ae + Bu(s) = B(u(s) - \bar{u})$. Thus, $|F(e, u(s))|_{2,P} = ((u(s) - \bar{u})'B'P'PB(u(s) - \bar{u}))^{1/2}$, and the bound (13) yields

$$\begin{aligned} &|\gamma(\tau) - e|_{2,P} \\ &\quad \leq \frac{1}{\eta} \max_{t\in[0,T]} ((u(t) - \bar{u})'B'P'PB(u(t) - \bar{u}))^{1/2} \end{aligned} \tag{17}$$

for all $\tau \in [0, T]$.

Of course, for linear systems the periodic solution corresponding to sinusoidal excitations is known explicitly in terms of the system's frequency response. Nevertheless, (17) seems to be new and provides considerable intuition: the bound on the distance between $\gamma(t)$ and e decreases when: the contraction rate η increases; the input channel B becomes "more orthogonal" to the matrix P in (16); or $\max_{t\in[0,T]} |u(t) - \bar{u}|$ decreases, that is, the periodic excitation becomes more similar to its mean. □

Example 2 The ribosome flow model (RFM) [23] is a nonlinear compartmental model describing the unidirectional flow of particles along a 1D chain of n sites using n nonlinear first-order differential equations. Recently, the RFM has been used to model and analyze the flow of ribosomes (the particles) along groups of codons (the sites) along the mRNA molecule during translation (see, e.g. [24], [25], [26], [21], [27], [28], [29], [30], [31]).

Consider the RFM with $n = 2$ and a *time-varying* initiation rate $u_0(t)$, that is,

$$\begin{aligned} \dot{x}_1 &= (1 - x_1)u_0 - \lambda_1 x_1(1 - x_2), \\ \dot{x}_2 &= \lambda_1 x_1(1 - x_2) - \lambda_2 x_2, \end{aligned} \tag{18}$$

where λ_1, λ_2 are positive constants. Suppose that $u_0(t) = \lambda_0 + \sin(2\pi t/T)$, with $\lambda_0 > 1$, $T > 0$, i.e. the initiation rate is a strictly positive periodic function with (minimal) period T. The state space here is $\Omega := [0, 1]^2$. The Jacobian of (18) is $J(t, x) = \begin{bmatrix} -u_0(t) - \lambda_1(1 - x_2) & \lambda_1 x_1 \\ \lambda_1(1 - x_2) & -\lambda_1 x_1 - \lambda_2 \end{bmatrix}$. The off-diagonal terms are non-negative for any $x \in [0, 1]^2$, so $\mu_1(J(t, x)) = \max\{-u_0(t), -\lambda_2\}$ for all $t \geq 0$ and all $x \in [0, 1]^2$. Thus, the system is contractive with respect to the ℓ_1 norm with contraction rate $\eta := \min\{\lambda_0 - 1, \lambda_2\} > 0$. Let $\gamma \in [0, 1]^2$ denote its unique, attracting, T-periodic solution. Entrainment in mRNA translation is important as biological organisms are often exposed to periodic excitations, for example the periodic cell division process. Proper biological functioning requires entrainment to such excitations [21].

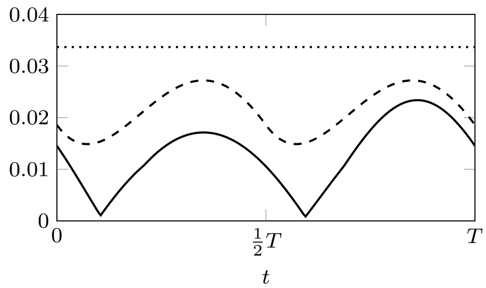

Fig. 1: RFM in Example 2. The error $|\gamma(t) - e|_1$ (solid line) and the bounds in Theorem 3: the bound (21) (dashed line) and the bound (22) (dotted line). These bounds can be obtained analytically for this example.

Let $\bar{u}_0 = \frac{1}{T}\int_0^T u_0(s)\,\mathrm{d}s = \lambda_0$ and consider the system

$$\begin{aligned} \dot{y}_1 &= \lambda_0(1 - y_1) - \lambda_1 y_1(1 - y_2), \\ \dot{y}_2 &= \lambda_1 y_1(1 - y_2) - \lambda_2 y_2. \end{aligned} \tag{19}$$

This system admits an equilibrium point

$$e = \begin{bmatrix} \frac{\lambda_0\lambda_1 - \lambda_0\lambda_2 - \lambda_1\lambda_2 + \sqrt{d}}{2\lambda_0\lambda_1} & \frac{\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2 - \sqrt{d}}{2\lambda_1\lambda_2} \end{bmatrix}' \in (0, 1)^2, \tag{20}$$

where $d := 4\lambda_0^2\lambda_1\lambda_2 + (\lambda_0\lambda_1 - \lambda_0\lambda_2 - \lambda_1\lambda_2)^2$.

Here,

$$F(e, u(s)) = \begin{bmatrix} (\lambda_0 + \sin(2\pi s/T))(1 - e_1) - \lambda_1 e_1(1 - e_2) \\ \lambda_1 e_1(1 - e_2) - \lambda_2 e_2 \end{bmatrix},$$

and since e is an equilibrium point of (19), $F(e, u(s)) = \begin{bmatrix} (1 - e_1)\sin(2\pi s/T) & 0 \end{bmatrix}'$. Thus, (12) yields

$$\begin{aligned} &|\gamma(\tau) - e|_1 \leq \\ &\quad (1 - e_1)\frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)} |\sin(2\pi s/T)|\,\mathrm{d}s \\ &\quad + (1 - e_1)\int_0^\tau e^{-\eta(\tau-s)} |\sin(2\pi s/T)|\,\mathrm{d}s \end{aligned} \tag{21}$$

for all $\tau \in [0, T]$. Furthermore, $|F(e, u(t))|_1 = (1 - e_1)|\sin(2\pi t/T)| \leq 1 - e_1$, so (13) implies the simpler yet more conservative bound

$$|\gamma(\tau) - e|_1 \leq (1 - e_1)/\eta, \quad \text{for all } \tau \in [0, T]. \tag{22}$$

Note that the bounds above can be computed analytically so that we obtain considerable explicit information on the periodic trajectory γ.

Fig. 1 illustrates the bounds on the periodic trajectory for the case $\lambda_0 = 4$, $\lambda_1 = 1/2$, $\lambda_2 = 4$, and $T = 2$. It may be seen that these bounds indeed provide a reasonable approximation for the ℓ_1 distance between the unknown periodic trajectory and the point e. □

Thm. 3 is based on averaging the excitation over a period, thus obtaining a constant input. Such an approximation is not always suitable. For example, when $u(t) = \sin(2\pi t/T)$ then $\bar{u} := \frac{1}{T}\int_0^T u(t)dt = 0$ for all T. This may obscure the effect of the *frequency* of the excitation in the derived bounds. The approach in the next subsection tries to overcome this using a different approximating system, namely, an LTI system that is excited by the original periodic input.

B. An LTI approximation

Theorem 4 *Consider the system*

$$\dot{x} = F(x, u), \tag{23}$$

where u is an m-dimensional periodic excitation with period $T > 0$. Suppose that the trajectories of (23) *evolve on a compact and convex state space $\Omega \subset \mathbb{R}^n$. Assume that for some vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ and the induced matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$,*

$$\mu\left(\frac{\partial F}{\partial x}(x, u(t))\right) \le -\eta < 0$$

for all $t \ge 0$, all $x \in \Omega$. Let $\gamma(t)$ be the unique, attracting, T-periodic orbit of (23) *in Ω.*

Suppose also that for the unforced dynamics, i.e. $\dot{x} = F(x, 0)$, there exists a locally stable equilibrium point $e \in \Omega$, and without loss of generality, that $e = 0$. Let $A := \frac{\partial F}{\partial x}(0, 0)$ and $B := \frac{\partial F}{\partial u}(0, 0)$, and consider the LTI approximating system

$$\dot{y} = Ay + Bu := G(y, u). \tag{24}$$

Pick $x_0, y_0 \in \Omega$ and let $\tau \ge 0$ be such that $y(t) \in \Omega$ for all $t \in [0, \tau]$ where $y(t)$ is the solution of (24) *with $y(0) = y_0$. Then*

$$|x(t) - y(t)| \le e^{-\eta t}|x_0 - y_0| + \int_0^t e^{-\eta(t-s)}|H(y(s), u(s))|\, \mathrm{d}s$$

for all $t \in [0, \tau]$ where $H(z, v) := F(z, v) - G(z, v)$. Moreover, let $\kappa(t)$ be the unique T-periodic trajectory of (24) *and assume that $\kappa(t) \in \Omega$ for all t. Then, for all $\tau \in [0, T]$,*

$$\begin{aligned}|\gamma(\tau) - \kappa(\tau)| &\le \frac{e^{-\eta\tau}}{1 - e^{-\eta T}} \int_0^T e^{-\eta(T-s)}|H(\kappa(s), u(s))|\, \mathrm{d}s \\ &\quad + \int_0^\tau e^{-\eta(\tau-s)}|H(\kappa(s), u(s))|\, \mathrm{d}s \qquad (25)\\ &\le \frac{1}{\eta} \max_{t\in[0,T]} |H(\kappa(t), u(t))|. \qquad (26)\end{aligned}$$

We emphasize again that the advantage of the bounds here is that the integrand depends on the difference between the vector fields F and G evaluated along the solution κ of the LTI system (24). Note that our assumptions imply that A is Hurwitz and thus, for any initial condition, $y(t)$ converges to the periodic trajectory $\kappa(t)$. In some cases, this solution can be written explicitly, and the integral can be computed explicitly. For example, if $u(t)$ is a complex exponential, then $\kappa(t)$ is also a complex exponential and can be easily computed using a Fourier transform. Then a bound on $|F(\kappa(t), u(t)) - G(\kappa(t), u(t))|$, $t \in [0, T]$, may be straightforward to establish, as in the example below.

Example 3 We again consider the RFM with $n = 2$ and the periodic initiation rate $u_0(t) := \lambda_0 + u(t)$, with $\lambda_0 > 1$ and $u(t) = \sin(2\pi t/T)$. Again, let e be the unique equilibrium of the system when the initiation rate is λ_0

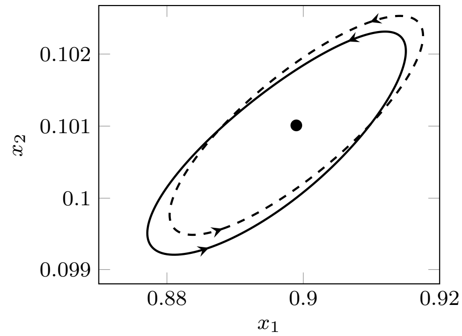

Fig. 2: RFM in Example 3. The equilibrium e for $\lambda_0 = 4$ is marked by a dot. The periodic trajectory $\gamma(t)$ of the RFM (solid line) and the periodic trajectory $\kappa(t)$ of the linearized system (dashed line) when $u_0(t) = 4 + \sin(2\pi t/T)$.

(see (20)). Let $\delta x := x - e$. Then the linearized system is $\dot{\delta x} = A\delta x + bu$, where

$$A = \begin{bmatrix} -\lambda_0 - \lambda_1(1 - e_2) & \lambda_1 e_1 \\ \lambda_1(1 - e_2) & -\lambda_1 e_1 - \lambda_2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 - e_1 \\ 0 \end{bmatrix}.$$

Note that $\mu_1(A) = \max\{-\lambda_0, -\lambda_2\} < 0$, so, in particular, A is Hurwitz. Thus, the approximating system is

$$\dot{y} = A(y - e) + bu =: G(y, u), \quad u(t) = \sin(2\pi t/T). \tag{27}$$

The difference between the vector fields evaluated along a solution of the y system is

$$F(y, \sin(2\pi t/T)) - G(y, \sin(2\pi t/T)) = \begin{bmatrix} \lambda_1(y_1 - e_1)(y_2 - e_2) - (y_1 - e_1)\sin(2\pi t/T) \\ -\lambda_1(y_1 - e_1)(y_2 - e_2) \end{bmatrix}.$$

Let $\hat{g}(s) := \begin{bmatrix} \hat{g}_1(s) \\ \hat{g}_2(s) \end{bmatrix} = (sI - A)^{-1}b$, and let $\kappa(t) : \mathbb{R} \to \mathbb{R}^2$ be the unique periodic trajectory of (27) defined for all $-\infty < t < \infty$. Then

$$\kappa(t) - e = \begin{bmatrix} |\hat{g}_1(j\omega)| \sin(\omega t + \angle\hat{g}_1(j\omega)) \\ |\hat{g}_2(j\omega)| \sin(\omega t + \angle\hat{g}_2(j\omega)) \end{bmatrix},$$

with $\omega := 2\pi/T$. The bound (26) yields

$$|\gamma(t) - \kappa(t)|_1 = \frac{1}{\eta}\left(2\lambda_1|\hat{g}_1(j\omega)||\hat{g}_2(j\omega)| + |\hat{g}_1(j\omega)|\right) \tag{28}$$

where $\eta := \min\{\lambda_0 - 1, \lambda_2\}$ as before. Note that the bound here depends on the frequency of the periodic excitation. The more exact bound in (25) can be computed numerically.

For the parameters $\lambda_1 = 1/2$, $\lambda_2 = 4$, and $T = 2$, Figure 2 shows the equilibrium point when $\lambda_0 = 4$, the periodic trajectory for the case when the initiation rate is $u_0(t) = 4 + \sin(2\pi t/T)$, and the periodic trajectory of the linearized system. Figure 3 illustrates the bounds from Theorem 4. It may be observed that these bounds provide a reasonable estimate of the error. □

The bound (28) has some interesting implications. For example, if $\hat{g}_1(j\omega) = 0$ for some ω then (28) implies that $\gamma(t) \equiv \kappa(t)$ for a sinusoidal excitation with frequency ω. Similarly, if $\lim_{\omega\to\infty} \hat{g}_1(j\omega) = 0$ then (28) implies that for

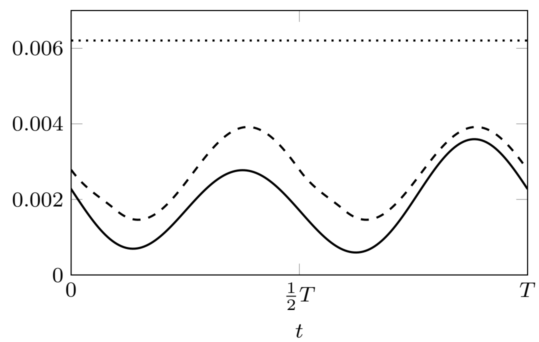

Fig. 3: RFM in Example 3. The error $|\gamma(t) - \kappa(t)|_1$ (solid line) and the bounds (25) (dashed lines) and (28) (dotted line).

a high frequency sinusoidal forcing term, γ will approach κ. Note that the conclusions on γ here are based on *properties of the LTI system*.

V. Conclusions

Contractive systems entrain to periodic excitations. Analyzing the corresponding periodic solution of the contractive system and its dependence on various parameters is an important theoretical question with many potential applications. We developed approximation schemes for this periodic solution using LTI systems and, using the ISS property of contractive systems, provided bounds on the approximation error. An important advantage of these bounds is that in some cases they can be computed explicitly.

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