The Strong Integral Input-to-State Stability Property in Dynamical Flow Networks

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Abstract—Dynamical flow networks serve as macroscopic models for, e.g., transportation networks, queuing networks, and distribution networks. While the flow dynamics in such networks follow the conservation of mass on the links, the outflow from each link is often non-linear due to, e.g., flow capacity constraints and simultaneous service rate constraints. Such non-linear constraints imply a limit on the magnitude of exogenous inflow that is able to be accommodated by the network before it becomes overloaded and its state trajectory diverges. This paper shows how the Strong integral Input-to-State Stability (Strong iISS) property allows for quantifying the effects of the exogenous inflow on the flow dynamics. The Strong iISS property enables a unified stability analysis of classes of dynamical flow networks that were only partly analyzable before, such as networks with cycles, multi-commodity flow networks and networks with non-monotone flow dynamics. We present sufficient conditions on the maximum magnitude of exogenous inflow to guarantee input-to-state stability for a dynamical flow network, and we also present cases when this sufficient condition is necessary. The conditions are exemplified on a few existing dynamical flow network models, specifically, fluid queuing models with time-varying exogenous inflows and multi-commodity flow models.

Index Terms—dynamical flow networks, input-to-state stability, transportation networks, queuing networks

I. Introduction

Dynamical flow networks serve as macroscopic models for physical network flows such as transportation networks [2], [3], [4] as well as non-physical processing networks such as queuing systems [5]. One common component for those networks is a limitation on the magnitude of exogenous inflow the networks can accommodate due to, e.g., maximum flow capacity on the roads or maximum processing capacity at a server. The dynamics of such networks is often non-linear, both due to the physical flow dynamics itself and saturation in the service rates. Thus, classical techniques such as linear system analysis are not enough to analyze these systems' stability properties.

In many of the aforementioned applications, the goal is to keep link densities or queues bounded. It is generally observed that this is possible as long as the exogenous inflow to the network stays below a certain threshold. The purpose of this paper is to formalize this observation as necessary and sufficient conditions for stability for a large class of dynamic flow

networks. The Strong Integral Input-to-State property (Strong iISS) was introduced in [6] for general dynamical systems to combine integral input-to-state stability (iISS) with input-to-state stability (ISS) for small inputs and to determine when the input is small enough to guarantee the latter. Although those two properties align naturally with the expected behavior of dynamical flow networks, the Strong iISS property has not been exploited for studying dynamical flow networks in a general setting with features such as cycles, multi-commodity flows, and non-monotone flow dynamics. In the context of dynamic flow networks, apart from having the desired property of guaranteeing stability when the exogenous inflow to the network is lower than a certain threshold, the Strong iISS property also imposes that if the exogenous inflow becomes zero at a certain time, the total mass in the network will also eventually converge to zero. For many applications, this is an essential property. For example, in transportation networks, a correct traffic signal control solution should trivially allow vehicles to leave the network eventually.

Existing literature on stability of dynamic flow networks often focuses on the class of networks with dynamics that exhibit a monotonicity property whereby state trajectories maintain a partial order on the state space [7], [8], [9], or a related mixed-monotonicity property [10]. In these cases, powerful results from monotone systems theory and contraction theory provide sufficient conditions for stability and, e.g., constructive methods for obtaining Lyapunov functions.

However, as noticed in [11], when extending flow network models to multi-commodity flows, the monotonicity property is usually lost. Another example of when the system's monotonicity property is lost is when a feedback controller can serve more than one queue simultaneously, and the service is split in proportion to the demand in all queues that are served simultaneously [12]. This situation is common in many applications, such as when controlling traffic signals in a transportation network. Other methods that have been proposed for considering specific classes of networks include using passivity theory [13] or constructing specific entropy-like Lyapunov functions [14], but a general framework for considering non-monotone dynamic flow networks remains elusive.

In this paper, we propose Strong iISS as another important tool, alongside monotone systems theory and contraction theory, for studying a large class of dynamic flow networks. The stability analysis in this paper partly relies on a special variant of sum-separable Lyapunov functions similar to those that have previously been combined with monotonicity properties, e.g., [9], [15]. In particular, the Lyapunov function is based on a transformation involving the inverse of the routing matrix for the network. This transformation has previously been utilized

to obtain monotonicity properties for tree-like flow networks in [16].

Preliminary results on using Strong iISS to study dynamical flow networks appeared in [1]. In this paper, we extend those results by considering both bounded and unbounded flow functions. We also show that, in the case of bounded flow functions, it is always optimal to normalize the stability condition by the capacity vector. Furthermore, we extend the stability analysis to network flow dynamics described by differential inclusions. This dynamic description is needed when there is a possibility that a link with no mass present can receive service, something that is common in, e.g., traffic network applications where several links may belong to the same service phase. We also extend the examples beyond those presented in [1].

The rest of the paper is organized as follows: The remainder of this section is devoted to introducing some basic notation that will be used throughout the paper. In Section II, we present the dynamical flow network model, together with a few general model assumptions. We also show that under those mild assumptions, dynamical flow networks are always iISS. In Section III we show that a dynamical flow network is Strong iISS. We present a sufficient condition on the exogenous inflow for the dynamical flow network to be ISS and show how this condition can be tightened when all the flow functions are bounded. We also present a case when the ISS condition is tight for a local network and an alternative bound on the growth rates of the state that differs from the standard Strong iISS bound. In the following section, Section IV, we extend the analysis to network flow dynamics where the outflow is described through differential inclusion. In Section V, we illustrate how the stability theory can be applied to existing models for dynamical flow networks, namely dynamical networks with time-varying exogenous inflows and multi-commodity flow networks. The paper is concluded with some ideas for future research.

A. Notation

We let $\mathbb{R}_+$ denote the non-negative reals. For a finite set $\mathcal{A}$, $\mathbb{R}_+^{\mathcal{A}}$ denote the set of non-negative vectors indexed by $\mathcal{A}$. Unless stated otherwise, for a vector $x \in \mathbb{R}^n$, we let $\|x\|$ denote the ℓ_1 norm. For vectors, all inequalities apply element-wise. For vectors $w, x \in \mathbb{R}_+^n$ such that $w > 0$, we introduce the weighted ℓ_1-norm as $\|x\|_w = w^T x$. The all-one vector is denoted by $\mathbb{1}$ with the appropriate dimension. We denote the indicator function for when a variable x is strictly positive as $\mathbb{1}_{(x>0)}$. A function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ if it is strictly increasing, $\mu(0) = 0$, and $\lim_{x \to +\infty} \mu(x) = +\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said be of class $\mathcal{K}\mathcal{L}$ if $\beta(0, t) = 0$ for all t, it is strictly increasing in x for each fixed t, and it is decreasing in t for each fixed x and $\lim_{t \to +\infty} \beta(x, t) \to 0$.

II. MODEL

We model a dynamical flow network as a directed multi-graph, i.e., in contrast to a directed graph there can exist multiple parallel links between two nodes, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the multiset of links. We will assume that the graph has no self-loops. For a link $e = (i, j) \in \mathcal{E}$, we let $\tau(e)$ denote the tail of the link, i.e., $\tau(e) = i$, and $\sigma(e)$ the head of the link, i.e., $\sigma(e) = i$. Moreover, we let $\mathcal{E}_v$ denote the subset of incoming links to node $v \in \mathcal{V}$, formally $\mathcal{E}_v = \{i \in \mathcal{E} \mid \sigma(i) = v\} \subset \mathcal{E}$.

In the flow network, mass flows along the links $\mathcal{E}$. Therefore, the network's state $x \in \mathcal{X} = \mathbb{R}_+^{\mathcal{E}}$ is the vector of masses on all the links in the network and $\mathcal{X}$ is the state space. We let $\lambda(t) \in \mathbb{R}_+^{\mathcal{E}}$ denote the vector of possible time-varying exogenous inflow to the links, and for links that can have no exogenous inflow, the corresponding elements in $\lambda(t)$ will be identically zero.

To model the mass propagation through the network, we introduce the routing matrix $R \in \mathbb{R}^{\mathcal{E} \times \mathcal{E}}$. The element R_{ij} in the routing matrix is the fraction of outflow from link i that will proceed to link j. Hence, $0 \leq R_{ij} \leq 1$. Moreover, due to conservation of mass it must hold that for all $i \in \mathcal{E}$, $\sum_j R_{ij} \leq 1$ where $1 - \sum_j R_{ij}$ is the fraction of mass that leaves the network after flowing out from link i. Since the routing matrix has to obey the network topology, $R_{ij} > 0$ only if $\sigma(i) = \tau(j)$.

Throughout the paper, we will make the assumption that the network is outflow connected, i.e., from every link in the network it is possible to find a path to a link where a fraction of the mass can leave the network after flowing out from this link.

Assumption 1: The routing matrix R is assumed to be out-flow connected, i.e., for every link $i \in \mathcal{E}$ there exists a path to a link $j \in \mathcal{E}$ such that $\sum_k R_{jk} < 1$. The existence of a path from link i to the link j can be equivalently expressed as that there exists an integer $\ell > 0$ such that $(R^\ell)_{ij} > 0$, i.e., there exists a path of length ℓ between the links.

Assumption 1 implies that the spectral radius of the routing matrix is less than one, and hence the matrix $(I - R^T)$ is invertible [17]. Its inverse can be computed through

$$(I - R^T)^{-1} = \sum_{k \geq 0} (R^T)^k = I + R^T + (R^T)^2 + \dots.$$

The outflow from each link is controlled by a state-dependent Lipschitz-continuous outflow function, which we denote $f_i : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}_+$ for every link $i \in \mathcal{E}$. We let $f(x)$ denote the vector of all flow functions, i.e., $f(x) = (f_i(x))_{i \in \mathcal{E}}$. In this setting we allow the outflow function to depend on the link's own state, which is common in applications where the outflow only depends on the mass transportation dynamics on the link itself. But we also allow for the outflow to be dependent on the state of the neighboring links or the whole network, something that is common in, e.g., queueing theoretic applications where the service has to be split among different links.

We will make the following assumption about the flow functions.

Assumption 2: We assume that the flow functions $f_i(x)$ are always non-negative and such that $f_i(x) = 0$ if and only if $x_i = 0$.

The "if" part of the assumption ensures that no mass can flow out from the link if there is no mass present on the link.

The "only if" part ensures that the flow functions are work-conservative, i.e., if there is a mass present on one link, at least some of it will flow out from the link. In Section IV, we will relax the "if" part of this assumption to adopt a broader set of outflow functions.

Remark 1: In contrast to some related work on dynamical flow networks, e.g., [8], we impose no monotonicity assumptions on the flow functions.

The dynamics then follows from the conservation of mass: the change of mass on each link equals the inflow from the upstream links combined with potential exogenous inflow minus the outflow from the link itself. That is,

$$\dot{x}_i = \lambda_i + \sum_{j \in \mathcal{E}} R_{ji} f_j(x) - f_i(x), \quad \forall i \in \mathcal{E},$$

which is expressed in vector form as

$$\dot{x} = \lambda - (I - R^T) f(x). \tag{1}$$

Remark 2: If the first part of the Assumption 2 holds, then with the dynamics (1) the state-space $\mathcal{X}$ is positively invariant. This since if $x_i = 0$ for some $i \in \mathcal{E}$, then $f_i(x) = 0$ and $\dot{x}_i = \lambda_i + \sum_{j \in \mathcal{E}} R_{ji} f_j(x) \geq 0$.

In many applications, the desired stability properties of the network flow dynamics (1) under Assumptions 1 and 2 are the following:

1) In the case of unbounded flow functions, the state should always stay bounded when the exogenous inflows are bounded. If the flow functions $f(x)$ are bounded from above, then for small enough λ, the state x should stay bounded.
2) If the exogenous inflow λ becomes zero, then the state x should eventually become zero as well.

The first property is often used to characterize the stability properties of dynamical flow networks, since the requirement of bounded states allows for, e.g., time-varying exogenous inflows. It should also be noted that when the flow functions are bounded, the dynamical flow network can not be ISS since there will exist exogenous inflows large enough to make the system unstable. The second property ensures that all the mass in the network will eventually leave the network, something that is desirable in, e.g., transportation network applications.

In the next section, we will see how these desired properties fit well with the Strong integral Input-to-State Stability (Strong iISS) property and show that the dynamical flow network dynamics (1) is Strong iISS.

III. STRONG iISS PROPERTY FOR DYNAMICAL FLOW NETWORKS

In the first part of this section, we show that dynamical flow networks satisfying Assumptions 1 and 2 are Strong iISS, which includes a sufficient condition on the exogenous inflows for the system to be ISS. We then show that this sufficient condition sometimes can be tightened by normalizing the flow capacities when all the flow functions are bounded. While it is possible to normalize the condition by any positive vector, we show that the capacity vector is the optimal choice. We

then show a special case when the sufficient condition is also necessary. In the last part of this section, we present an alternative bound to the standard Strong iISS bound on the state's growth rate.

A. Dynamical Flow Networks are Strong iISS

Let us first recall the definition of Strong iISS from [6].

Definition 1 (Strong iISS, [6, Def. 2]): A dynamical system $\dot{x} = f(x, u)$ is said to be

1) *Integral Input-to-State Stable (iISS)* if there exists a function $\beta \in \mathcal{KL}$, and functions $\mu_1, \mu_2, \mu \in \mathcal{K}_\infty$ such that for all $x(0) \in \mathbb{R}_+^{\mathcal{E}}$ and all $t \geq 0$ the solution to $\dot{x} = f(x, u)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \mu_1 \left(\int_0^t \mu_2(\|u(s)\|) \mathrm{d}s \right).$$

2) *ISS with respect to small inputs* if there exists a function $\beta \in \mathcal{KL}$, a function $\mu \in \mathcal{K}_\infty$, and a constant input threshold $U > 0$ such that when

$$\operatorname*{ess\,sup}_{t \geq 0} \|u(t)\| < U,$$

for all $x(0) \in \mathbb{R}_+^{\mathcal{E}}$ and all $t \geq 0$ the solution to $\dot{x} = f(x, u)$ satisfies the bound

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \mu \left(\operatorname*{ess\,sup}_{t \geq 0} \|u(t)\| \right).$$

If a system is ISS with respect to small inputs and the above bound holds for all inputs satisfying some condition $\mathcal{C}$, we also say the system is *ISS with respect to inputs satisfying $\mathcal{C}$*.

3) *Strong iISS* if the system is both iISS and ISS with respect to small inputs, i.e., both 1) and 2) hold true.

The first part of the definition states that the size of the state is limited by the integral of the input and is the classical integral Input-to-State Stability (iISS) requirement [18]. The second part of the definition implies that the state is always bounded when the input is small enough, and in the following, we will characterize input thresholds for dynamical flow networks such that the above state bound holds. In the case when the definition is valid for any choice of U, the system has the classical Input-to-State Stability (ISS) property. It should be noted that when there exists a time $t' \geq 0$ such that the input $u(t) = 0$ for all $t \geq t' \geq 0$, then the state will eventually converge to zero. Hence, the second part of Definition 1 captures the desired stability properties mentioned in Section II.

Next, we will prove that dynamical flow networks are Strong iISS.

Theorem 1: A dynamical flow network (1) satisfying Assumption 1 and 2 is strong iISS. In particular, it is ISS with respect to inputs satisfying

$$\operatorname*{ess\,sup}_{t \geq 0} \sum_{i \in \mathcal{E}} a_i(t) < \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}} f_i(x). \tag{2}$$

where $a(t) = (I - R^T)^{-1} \lambda(t)$.

Proof: Introduce the Lyapunov candidate

$$V = \mathbb{1}^T (I - R^T)^{-1} x \,. \tag{3}$$

Since $x \geq 0$ and $(I - R^T)^{-1} = \sum_{k \geq 0} (R^T)^k$, the Lyapunov candidate (3) satisfies $V(x) \geq 0$ and $\overline{V}(x) = 0$ if and only if $x = 0$. Moreover,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} f(x) = \mathbb{1}^T (I - R^T)^{-1} (\lambda - (I - R^T) f(x))$$

$$= \sum_{i \in \mathcal{E}} a_i(t) - \sum_{i \in \mathcal{E}} f_i(x) \,. \tag{4}$$

Let $\gamma(x) = x$ and $W(x) = \sum_{i \in \mathcal{E}} f_i(x)$. Clearly, $\gamma \in \mathcal{K}_\infty$, and

$$\gamma(\|(I - R^T)^{-1} \lambda(t)\|) = \sum_{i \in \mathcal{E}} a_i(t) \,.$$

Eq. (4) can be rewritten

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} f(x) = -W(x) + \gamma(\|(I - R^T)^{-1} \lambda(t)\|) \,.$$

Due to Assumption 2, $W(x)$ is positive definite. Now, applying [6, Theorem 1] gives the bound. ∎

Remark 3: The bound (2) in Theorem 1 captures the physical properties of the network. In the case when some or all the flow functions are bounded by some capacity constraints, the right-hand side of (2) will attain the minimum of these capacity constraints and hence impose a limit on the exogenous inflows.

Although the bound may be conservative, we will later on exemplify when the bound is tight.

B. The Strong iISS Property for Bounded Flow Functions

In the case when all the flow functions are bounded such that outflow capacity from each link is given by $c_i = \sup_{x \geq 0} f_i(x)$, it is sometimes possible to state a less conservative condition than (2) on the set of inputs that results in bounded state trajectories. Let $c = (c_i)_{i \in \mathcal{E}}$ be the vector of all capacities, $\bar{c} = (1/c_i)_{i \in \mathcal{E}}$ and $C = \text{diag}(c)$. Moreover, for all links $i \in \mathcal{E}$, introduce the normalized flow functions

$$\tilde{f}_i(x) = \frac{f_i(x)}{c_i} \,,$$

and let $\tilde{f}(x) = (\tilde{f}_i(x))_{i \in \mathcal{E}}$ be the vector of all flow functions. The dynamics (1) can then equivalently be written as

$$\dot{x} = \lambda - (I - R^T) C \tilde{f}(x) \,.$$

For dynamical flow networks with bounded flow functions, we obtain the following corollary of Theorem 1.

Corollary 1: Consider a dynamic flow network satisfying the hypotheses of Theorem 1. If, in addition, $c_i = \sup_{x \geq 0} f_i(x)$ is finite for all i, then the system is ISS with respect to small inputs satisfying

$$\operatorname*{ess\,sup}_{t \geq 0} \sum_{i \in \mathcal{E}} \frac{a_i(t)}{c_i} < \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}} \tilde{f}_i(x) \,, \tag{5}$$

where $a(t) = (I - R^T)^{-1} \lambda(t)$.

Proof: The proof follows the same way as the proof for Theorem 1, but with the Lyapunov candidate

$$V(x) = \mathbb{1}^T C^{-1} (I - R^T)^{-1} x \,.$$
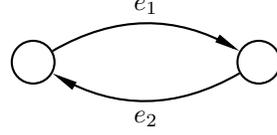
Fig. 1. The two node network used in Example 1 to illustrate a case when the bound for ISS in Theorem 1 is tight.

∎

The condition (5) in Corollary 1 can be less conservative than its counterpart (2) in Theorem 1, as the following example shows.

Example 1: Consider the two-link, two-node network in Figure 1. Let $f_1(x_1) = c_1(1 - \exp(-x_1))$ and $f_2(x_2) = c_2(1 - \exp(-x_2))$ with $c_1 = 1$ and $c_2 \gg 1$. Let the routing matrix R be such that $R_{1,2} = 0.9$ and $R_{2,1} = 1$. Then $a_1 = 10\lambda_1 + 10\lambda_2$ and $a_2 = 9\lambda_1 + 10\lambda_2$.

Using the condition in Theorem 1, the sufficient condition (2) becomes

$$19\lambda_1 + 20\lambda_2 < \min(c_1, c_2) = 1 \,,$$

but by utilizing the fact that the functions are bounded, the condition (5) in Corollary 1 becomes

$$\frac{\lambda_1 + \lambda_2}{c_1} + \frac{\lambda_1 + \lambda_2}{c_2} < 1$$

which guarantees bounded state trajectories for larger exogenous inflows when c_2 is large. In fact, when $c_2 \to +\infty$, the condition above becomes

$$\lambda_1 + \lambda_2 < 1$$

which is arbitrary close to the necessary condition that $\lambda_1 + \lambda_2 \leq 1$.

Remark 4: It is not always the case that the bound in Corollary 1 is less conservative than the bound in Theorem 1. For example, in a network with one link with exogenous inflow λ_1 such that $c_1 = \liminf_{x_1 \to +\infty} f_1(x_1)$, the condition (5) reads $\lambda_1 < c_1^2$ and is hence only less conservative when $c_1 > 1$.

The previous example raises the question of whether, in the case of bounded flow functions, another choice of normalizing vector instead of the capacity vector c can yield a more relaxed sufficient condition. To answer this question, we start by observing that for the case when all the flow functions are such that

- the flow only depends on the mass on the link itself, i.e., $\frac{\partial}{\partial x_j} f_i(x) = 0$ for all $j \neq i \in \mathcal{E}$, and
- the flow attains it maximum in its limit, i.e., $\lim_{x_i \to +\infty} f_i(x) = c_i$ for all $i \in \mathcal{E}$,

the condition (5) in Corollary 1 reads

$$\operatorname*{ess\,sup}_{t \geq 0} \sum_{i \in \mathcal{E}} \frac{a_i(t)}{c_i} \leq 1 \,.$$

The following proposition shows that in this case, the capacity is the optimal choice of a normalizing vector.

Proposition 1: Given vectors $a \in \mathbb{R}^{\mathcal{E}}$ and $c \in \mathbb{R}_+^{\mathcal{E}}$ where $c > 0$, suppose that there exists a strictly positive vector $b \in \mathbb{R}_+^{\mathcal{E}}$, $b > 0$ such that

$$\sum_{i \in \mathcal{E}} \frac{a_i}{b_i} \leq \min_{i \in \mathcal{E}} \frac{c_i}{b_i}. \tag{6}$$

Then the inequality (6) also holds for the choice $b = c$, i.e.,

$$\sum_{i \in \mathcal{E}} \frac{a_i}{c_i} \leq 1.$$

Proof: Let

$$i^* = \arg\min_{i \in \mathcal{E}} \frac{c_i}{b_i},$$

then inequality (6) is equivalent to

$$\sum_{i \in \mathcal{E}} \frac{a_i}{b_i} \frac{b_{i^*}}{c_{i^*}} \leq 1.$$

But since $\frac{c_i^*}{b^*} \leq \frac{c_i}{b_i}$ for all $i \in \mathcal{E}$, it also holds that $\frac{b_i}{c_i} \leq \frac{b_{i^*}}{c_{i^*}}$ for all $i \in \mathcal{E}$ and hence

$$1 \geq \sum_{i \in \mathcal{E}} \frac{a_i}{b_i} \frac{b_{i^*}}{c_{i^*}} \geq \sum_{i \in \mathcal{E}} \frac{a_i}{b_i} \frac{b_i}{c_i} = \sum_{i \in \mathcal{E}} \frac{a_i}{c_i}.$$

∎

C. Necessary Condition for Local Networks

While the condition (5) in Corollary 1 and its counterpart (2) in Theorem 1 usually only are sufficient for ensuring bounded state, there are special cases when the condition becomes arbitrarily close to being necessary. One such example is a local network, i.e., a network where all the links point to one node, as shown in Figure 2.

Example 2: For a local network with constant inflows λ, the dynamics in (1) simplifies to

$$\dot{x}_i = \lambda_i - c_i f_i(x), \quad \forall i \in \mathcal{E}_v$$

and the solution satisfies

$$x(t) = x(0) + \lambda t - C \int_0^t f(x(s)) \mathrm{d}s.$$

By multiplying both sides by $\mathbb{1}^T C^{-1}$, we have

$$\mathbb{1}^T C^{-1}(x(t) - x(0)) = \int_0^t \left(\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} - f_i(x(s)) \right) \mathrm{d}s,$$

where the right hand side goes to infinity as $t \to +\infty$ if

$$\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} > \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}_v} f_i(x).$$

Since $\mathbb{1}^T C^{-1}$ is a strictly positive vector and $x(t) > 0$ for all $t \geq 0$, it follows that $\sum_{i \in \mathcal{E}} x_i(t) \to +\infty$, which shows that condition (5) with a non-strict inequality, i.e.,

$$\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} \leq \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}_v} f_i(x), \tag{7}$$

is a necessary condition for the states to be bounded.
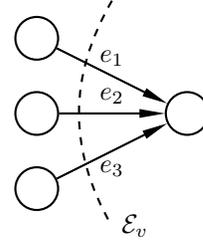
Fig. 2. Example of a local network, i.e., a network where all the links point towards a single node.

Remark 5: In the case when the flow functions are bounded from above such that $c_i = \liminf_{x_i \to +\infty} f_i(x_i)$, the condition (7) reads

$$\sum_{i \in \mathcal{E}_v} \frac{\lambda_i}{c_i} \leq 1.$$

This condition is similar to resource utilization conditions in processor scheduling [19, Theorem 4.2] and queuing networks [20, p. 9].

D. Alternative Bound on the Growth Rate

The strong iISS condition implied by Theorem 1 provides a bound on the norm of the state characterized by $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions not explicitly constructed. Instead, the following bound applies for each link and follows from the observation that the total amount of mass in the dynamical flow network will always be bounded by its initial state and the amount of exogenous inflow to the network. This bound does not require Assumption 2.

Proposition 2: Consider a dynamical flow network (1) satisfying Assumption 1 and let $a(t) = (I - R^T)^{-1} \lambda(t)$. Then

$$x_i(t) \leq \int_0^t a_i(s) \mathrm{d}s + \xi_i, \quad \forall i \in \mathcal{E},$$

where $\xi = (I - R^T)^{-1} x(0)$.

Proof: Let $\hat{x} = (I - R^T)^{-1} x$. Then

$$\dot{\hat{x}} = (I - R^T)^{-1} \lambda(t) - f(x) = a(t) - f(x)$$

and $\hat{x}(0) = (I - R^T)^{-1} x(0)$. Since $f(x) \geq 0$, it holds that $\dot{\hat{x}}_i \leq a_i(t)$ for all $i \in \mathcal{E}$, and hence

$$\hat{x}_i(t) \leq \int_0^t a_i(s) \mathrm{d}s + \hat{x}_i(0), \quad \forall i \in \mathcal{E}. \tag{8}$$

Observe that $\hat{x}(t) \geq 0$ for all $t \geq 0$ because $(I - R^T)^{-1} = \sum_{k \geq 0} (R^T)^k$ has all elements non-negative and both $\lambda \geq 0$ and $x(0) \geq 0$.

By transforming back to x, i.e., $x = (I - R^T)\hat{x}$, it then holds for each $i \in \mathcal{E}$ that

$$x_i(t) = \hat{x}_i(t) - \sum_{j \in \mathcal{E}} R_{ji} \hat{x}_j(t) \leq \hat{x}_i(t) \leq \int_0^t a_i(s) \mathrm{d}s + \hat{x}_i(0).$$

∎

Remark 6: In (8), the term $\int_0^t a_i(s) \mathrm{d}s$ is the total mass that can possibly reach link $i \in \mathcal{E}$ from outside the network, and the term $\tilde{x}_i$ indicates how much mass can reach link i from inside the network.

IV. DIFFERENTIAL INCLUSION DYNAMICS

In the previous sections we assume that $f_i(x) = 0$ when $x_i = 0$. However, there are applications, e.g., queuing theory and traffic signal control, where this assumption does not hold. For example, in traffic signal control, it can happen that several lanes belong to the same service phase, and then traffic present in one of the lanes will trigger the controller to serve all the lanes in the phase, including those that are empty. However, all the states will still stay non-negative.

To model this, we introduce a new state, the actual outflow $z \in \mathbb{R}_+^{\mathcal{E}}$. The flow dynamics now reads

$$\dot{x} = \lambda - (I - R^T)z \,. \tag{9}$$

The outflow is non-negative and is always upper bounded by the flow functions $f(x)$, i.e.,

$$0 \le z \le f(x) \,.$$

Moreover, we assume that if there is mass present on the link, i.e., $x_i > 0$ for a link $i \in \mathcal{E}$, the amount of outflow will be equal to the one given by the flow function, i.e., $z_i = f_i(x)$. This additional constraint is expressed as

$$x^T(z - f(x)) = 0 \,. \tag{10}$$

The dynamics (9)–(10) is a differential inclusion since when $x_i = 0$ for some link $i \in \mathcal{E}$, the actual outflow z_i can be anything between 0 and $f_i(x)$.

It can be shown that the dynamics in (9)–(10) is well-posed, i.e., for a given initial state $x(0)$, there exists a unique solution [21].

The following theorem, which generalizes the previously stated Theorem 1, can be used to analyze dynamical flow networks where the flows are determined through differential inclusion.

Theorem 2: A differential inclusion flow dynamics (9)–(10) satisfying Assumption 1, and $f(x) \ge 0$, and $f_i(x) > 0$ when $x_i > 0$ for all $i \in \mathcal{E}$ is strong iISS. In particular, it is ISS with respect to inputs satisfying

$$\operatorname*{ess\,sup}_{t \ge 0} \sum_{i \in \mathcal{E}} a_i(t) < \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}} \mathbb{1}_{(x_i > 0)} f_i(x) \,,$$

where $a(t) = (I - R^T)^{-1}\lambda(t)$.

Proof: Introduce the Lyapunov candidate

$$V(x) = \mathbb{1}^T(I - R^T)^{-1}x \,.$$

Since $(I - R^T)^{-1} = I + R^T + (R^T)^2 + \dots$ and $x \ge 0$, it holds that $V(x) \ge 0$ and $V(x) = 0$ if and only if $x = 0$. Moreover,

$$\begin{aligned}
\frac{dV}{dt} &= \mathbb{1}^T(I - R^T)^{-1}\dot{x} \\
&= \sum_{i \in \mathcal{E}} a_i(t) - \sum_{i \in \mathcal{E}:x_i > 0} f_i(x) - \sum_{i \in \mathcal{E}:x_i = 0} z_i \\
&\le \sum_{i \in \mathcal{E}} a_i(t) - \sum_{i \in \mathcal{E}} \mathbb{1}_{(x_i > 0)} f_i(x) \,.
\end{aligned}$$

Now, following the same methodology as in the proof of Theorem 1, but with $W(x) = \sum_{i \in \mathcal{E}} \mathbb{1}_{(x_i > 0)} f_i(x)$ instead (which is positive definite, since $f_i(x)$ is assumed to be strictly positive when $x_i > 0$), [6, Theorem 1] can again be applied to obtain the result. ∎
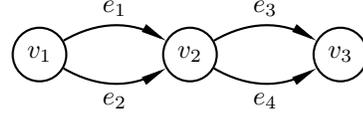
Fig. 3. The network for the example in Section V-A.
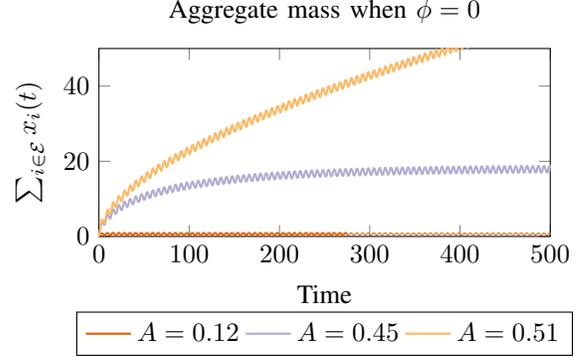
Fig. 4. The aggregate mass on all the links for the example in Section V-A. Although the sufficient condition only ensures stability for $A = 0.125$, the system is stable for $A = 0.45$ too. For $A = 0.51$, the trajectory diverges.

V. NUMERICAL EXAMPLES

In this section we present two examples, both illustrative of how the theory in this paper allows to analyze the stability of dynamical flow networks for broader classes of dynamics than was previously possible.

A. Time-Varying Inflows

Consider the network in Figure 3. Suppose that there are only exogenous inflows to the first two links, i.e., $\lambda_3 = \lambda_4 = 0$ and let $R_{1,4} = R_{1,3} = 0.5$ and $R_{2,4} = 1$. Suppose that each node splits the outflow from the incoming links according to

$$f_i(x) = \frac{x_i}{\sum_{j \in \mathcal{E}_v} x_j + 1} \,, \quad \forall i \in \mathcal{E}_v \,, \forall v \in \mathcal{V} = \{v_2, v_3\} \,.$$

For static inflows, the sufficient and necessary condition for bounded state presented in [22] is $\lambda_1 + \lambda_2 < 1$.

Theorem 1 makes it possible to ensure stability for time-varying inflows such that the sufficient condition to ensure bounded state is now

$$\operatorname*{ess\,sup}_{t \ge 0} \lambda_1(t) + \lambda_2(t) < \frac{1}{2} \,. \tag{11}$$

However, this condition is generally only sufficient. To illustrate this, we let

$$\lambda_1(t) = A(\sin(t) + 1), \quad \lambda_2(t) = A(\sin(t + \phi) + 1) \,,$$

and we consider several choices of A and ϕ. When $\phi = 0$, the sufficient condition in (11) is equivalent to $A < 0.125$. However, as can be seen in Figure 4, the state remains bounded for $A = 0.45$ but becomes unbounded for $A = 0.51$, which illustrates the sufficiency of the condition. By instead letting $\phi = \pi$, the sufficient condition in (11) is now equivalent to $A < 0.25$. The aggregate mass trajectories are shown in Figure 5 for the different choices of A when $\phi = \pi$.
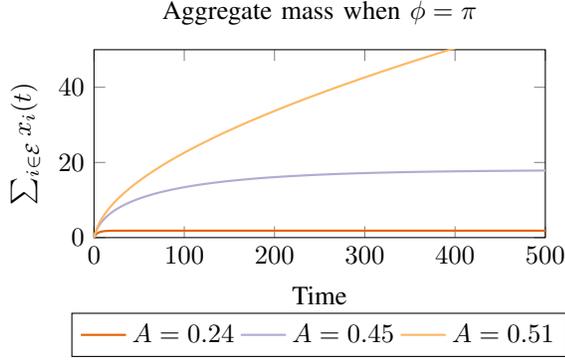
Fig. 5. The aggregate mass on all the links for the example in Section V-A. In this case, with $\phi = \pi$, the sufficient condition ensures stability for $A = 0.24$. For $A = 0.45$ the sufficient condition is not stabilized, but the trajectory remains bounded. For $A = 0.51$ the trajectory diverges.
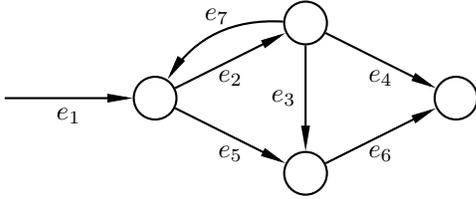
Fig. 6. The multi-commodity flow network used in the example in Section V-B.

B. Multi-Commodity Flows

Although the analysis in this note is done for single commodity flows, it can be extended to multi-commodity flows, i.e., dynamical flow networks where different commodities share the same network, but differ in routing. In particular, the technique can be used to accommodate any finite number of commodities, but for the purpose of this example, assume that we have two commodities, which we will denote A and B. Let $\lambda^A, \lambda^B \in \mathbb{R}_+^{\mathcal{E}}$ denote the exogenous inflow of the respective commodity. Each commodity is assumed to have its own routing matrix, which we denote R^A and R^B, where both routing matrices are assumed to satisfy Assumption 1 individually. As state space, we now need to keep track of the mass of each commodity on every link in the network, i.e., the state is (x^A, x^B) with $x^A, x^B \in \mathbb{R}_+^{\mathcal{E}}$. We let the vector $x \in \mathbb{R}_+^{\mathcal{E}}$ be the aggregate mass on each link, i.e., $x_i = x_i^A + x_i^B$ for every link $i \in \mathcal{E}$. Under the assumption that the commodities are perfectly mixed and travel with the same aggregate flow dynamics, the dynamics for the flow network with bounded outflow functions becomes

$$\dot{x}^A = \lambda^A - (I - (R^A)^T)C\mathrm{diag}\left(\left(\frac{x_i^A}{x_i}\right)_{i \in \mathcal{E}}\right)f(x),$$

$$\dot{x}^B = \lambda^B - (I - (R^B)^T)C\mathrm{diag}\left(\left(\frac{x_i^B}{x_i}\right)_{i \in \mathcal{E}}\right)f(x).$$

This model has previously been used to study road traffic flows where different commodities have different routing objectives in [23]. Different from the stability results presented in that paper, here, we allow the network to contain cycles.

Consider the network in Figure 6. Let the outflow functions for each link be $f_i(x_i) = 6(1 - e^{-x_i})$. The non-

TABLE I
THE NON-ZERO ELEMENTS IN THE ROUTING MATRICES

	Commodity A	Commodity B
$R_{1,2}$	0.6	0.7
$R_{1,5}$	0.4	0.3
$R_{2,7}$	0.1	0.3
$R_{2,3}$	0.3	0.4
$R_{2,4}$	0.6	0.3
$R_{3,6}$	1	1
$R_{5,6}$	1	1
$R_{7,2}$	0.5	0.3
$R_{7,5}$	0.5	0.7

zero elements in the routing matrix for each commodity are specified in Table I. We observe that for both commodities $k \in \{A, B\}$, it holds that $\sum_j R_{4,j}^k = \sum_j R_{6,j}^k = 0$, and the paths e_1, e_2, e_7, e_3, e_6, e_1, e_5, e_6, and e_1, e_2, e_4 exists for both commodities. Hence both the routing matrices satisfy Assumption 1.

Define

$$a^A(t) = (I - (R^A)^T)^{-1}\lambda^A(t),$$
$$a^B(t) = (I - (R^B)^T)^{-1}\lambda^B(t).$$

By using the Lyapunov function

$$V(x) = \mathbb{1}^T C^{-1}(I - (R^A)^T)^{-1}x^A + \mathbb{1}^T C^{-1}(I - (R^B)^T)^{-1}x^B,$$

and the same theory as in the proof of Theorem 1 and Corollary 1, we obtain the following sufficient condition for stability of the multi-commodity dynamics:

$$\underset{t \geq 0}{\mathrm{ess\,sup}} \sum_{i \in \mathcal{E}} \frac{a_i^A(t) + a_i^B(t)}{c_i} < \liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}} f_i(x). \quad (12)$$

In this example, note that $\liminf_{\|x\| \to +\infty} \sum_{i \in \mathcal{E}} f_i(x_i) = 1$. If we let $\lambda_1^A = 1$, $\lambda_1^B = 0.7$ and all other elements of λ^A, λ^B be zero, the sufficient condition in (12) is satisfied. The trajectory for each commodity is shown in Figure 7, with the initial state $x_i^A(0) = 0.3$ and $x_i^B(0) = 0.5$ for all $i \in \mathcal{E}$.

VI. CONCLUSIONS

In this note, we have shown how the Strong integral Input-to-State Stability (Strong iISS) framework is particularly suitable for studying the stability of dynamical flow networks. We established sufficient conditions on the exogenous inflows for dynamical flow networks to be stable, i.e., ensure that the state remains bounded, and showed that the condition is also necessary for certain types of networks. We also showed how the conditions can be applied to existing dynamical flow network models and provided stability assurance in settings not covered in prior literature.

A future research direction is to explore if the theory of Strong iISS leads to alternative tighter bounds in certain settings, e.g., by considering the time-averaged exogenous inflow or dividing the network into several sub-networks. Another topic for future research is also to explore how the Strong iISS property can be extended or adapted to dynamical flow networks where the storage is limited, such as certain models for traffic flows.

Commodity $k = A$

Commodity $k = B$

Legend:
$x_1^k(t)$ $x_2^k(t)$ $x_3^k(t)$ $x_4^k(t)$
$x_5^k(t)$ $x_6^k(t)$ $x_7^k(t)$
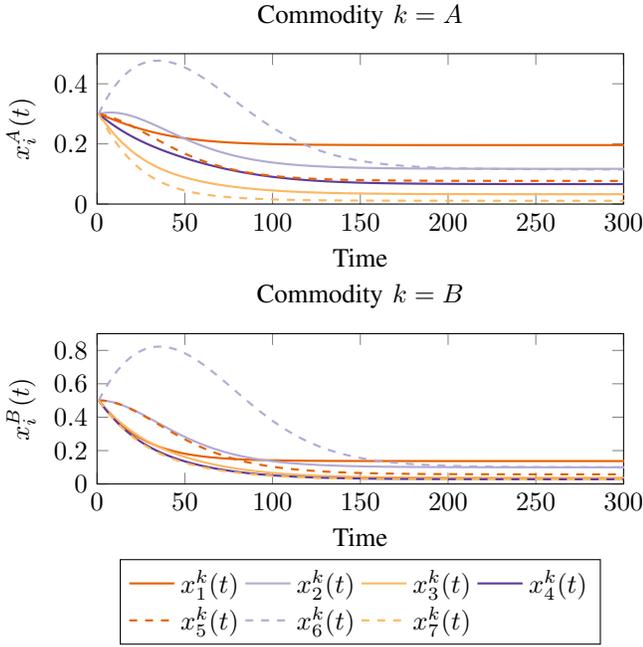
Fig. 7. The trajectories for commodity A and commodity B respectively in the example in Section V-B.

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